ANDERSON LOCALIZATION IN THE MULTI-PARTICLE TIGHT-BINDING MODEL AT LOW ENERGIES OR WITH WEAK INTERACTION

TRÉSOR EKANGA*

Abstract. We consider the multi-particle lattice Anderson model with an i.i.d. random external potential and a short-range interaction. Using the multi-particle multiscale analysis (MPMSA) developed by Chulaevsky and Suhov [6], we prove spectral localization for such Hamiltonians at low energies under the assumption of log-Hölder continuity of the marginal probability distribution of the random potential. Under a stronger assumption of Hölder continuity, Anderson localization for such systems at low energies was established earlier by Aizenman and Warzel (2009) [1] with the help of the multi-particle Fractional-Moment Method.

1. Introduction

Multi-particle Anderson localization theory is a relatively recent direction in the spectral theory of random operators. The general structure of an N-particle Hamiltonian of a lattice quantum system with interaction is as follows:

$$\mathbf{H}^{(N)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^{N} V(x_j, \omega) + \mathbf{U}(x_1, \dots, x_N) \tag{1.1}$$

where $\mathbf{\Delta}$ is the nearest-neighbor lattice Laplacian in $(\mathbb{Z}^d)^N \simeq \mathbb{Z}^{Nd}$, $V\colon \mathbb{Z}^d \times \Omega \to \mathbb{R}$ is a random field on $\mathbb{Z}^d$ relative to some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $x_1, \dots, x_N \in \mathbb{Z}^d$, and $\mathbf{U}\colon (\mathbb{Z}^d)^N \to \mathbb{R}$ is the potential energy of inter-particle interaction.

The first mathematically rigorous results have been obtained by Chulaevsky and Suhov [5, 6] with the help of the Multi-Scale Analysis (MSA) and by Aizenman and Warzel [1] who used the Fractional-Moment Method (FMM). In both cases, it was assumed that the random potential field is i.i.d., and the interaction $\mathbf{U}$ has finite range. In [6], the multi-particle spectral localization was proven for strongly disordered systems. The method of [1, 2] allows to establish dynamical localization in the following situations:

- Strongly disordered systems — for all energies.
- Weakly disordered systems — for "extreme" energies.
- Multi-particle systems with weak interaction, under the assumption that the single-particle subsystems are localized in absence of interaction.

Due to technical requirements of the FMM, it was assumed in [1] that the i.i.d. random field V has Hölder-continuous common marginal distribution.

Date: January 11, 2012.
2010 *Mathematics Subject Classification.* Primary 47B80, 47A75. Secondary 35P10.
Key words and phrases. Anderson localization, random operators, multi-particle, low energies.

In the present paper, we show, using Multi-Particle Multi-Scale Analysis (MPMSA), that this assumption can be relaxed to log-Hölder continuity of the marginal distribution: see assumption (**P2**) below and assertion (B) of Theorem 1.

On the other hand, under a more restrictive condition — assumption (**P1**) given below, one can apply a simpler multi-particle scaling procedure based on a recent eigenvalue concentration bound proven in [4] and establish exponential spectral localization along with more efficient intermediate bounds on the decay of eigenfunctions in finite cubes: see assertion (A) of Theorem 1.

Our main results are Theorem 1 (multi-particle localization at low energies), Theorem 2 and Corollary 1 (stability of multi-particle localization for weak interaction).

Section 2 is devoted to the description of the multi-particle model and to the main statements. Assertion (A) of Theorem 1 is proven under assumption (**P1**) in Sections 3 and 4. The parts of Theorem 1 under assumption (**P2**) are proven in Section 5. Finally, Theorem 2 and Corollary 1 are proven in Section 6.

2. The model and the main results

2.1. **Basic notations.** We are interested in the N-particle hamiltonian (1.1) under some assumptions on $\mathbf{\Delta}$, V and $\mathbf{U}$. In what follows $N \geq 2$ is fixed. Since multiscale analysis applied to this hamiltonian requires also the consideration of hamiltonians for n-particle subsystems for any $1 \leq n \leq N$, we introduce notations for such n.

2.1.1. *The Laplacian in $\mathbb{Z}^\nu$.* We generally equip $\mathbb{Z}^\nu$ with the max-norm $|\cdot|$ defined by

$$|x| = \max\{|x_1|, \dots, |x_\nu|\},$$

for $x = \big(x_1, \dots, x_\nu\big) \in \mathbb{Z}^\nu$, but, in the definition of the laplacian we will use another norm defined by

$$|x|_1 = |x_1| + \cdots + |x_\nu|.$$

We denote by Δ the ν-dimensional lattice nearest-neighbor Laplacian acting on the Hilbert space $\ell^2(\mathbb{Z}^\nu)$ by

$$(\Delta\Psi)(x) = \sum_{\substack{\mathbf{y}\in\mathbb{Z}^\nu\\ |\mathbf{y}-\mathbf{x}|_1=1}} \big(\Psi(y) - \Psi(x)\big) = -2\nu\Psi(x) + \sum_{\substack{\mathbf{y}\in\mathbb{Z}^\nu\\ |\mathbf{y}-\mathbf{x}|_1=1}} \Psi(y).$$

Note that Δ is bounded and $-\Delta$ is nonnegative.

2.1.2. *The n-particle Hamiltonian.* Let $n \geq 1$ and $d \geq 1$ be two integers. A configuration of n distinguishable quantum particles in the lattice $\mathbb{Z}^d$ is represented by a lattice vector

$$\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{Z}^d)^n \cong \mathbb{Z}^{nd},$$

where $x_j = (x_j^{(1)}, \dots, x_j^{(d)}) \in \mathbb{Z}^d$, $j = 1, \dots, n$. We will consider a random Hamiltonian $\mathbf{H}^{(n)}$ of n particles in $\mathbb{Z}^d$ of the form

$$\mathbf{H}^{(n)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^{n} V(x_j, \omega) + \mathbf{U}, \tag{1.1}$$

acting in the Hilbert space $\mathcal{H}^{(n)} = \ell^2(\mathbb{Z}^{dn})$. It is easy to see that both norms are compatible with the identification $\mathbb{Z}^{dn} \simeq (\mathbb{Z}^d)^n$, i.e.,

$$|\mathbf{x}| = \max_{1\leq j\leq n} |x_j|,$$

$$|\mathbf{x}|_1 = |x_1|_1 + \cdots + |x_n|_1.$$

Hence we have

$$\mathbf{\Delta} = \sum_{j=1}^{n} \underbrace{\mathbf{I}_{\ell^2(\mathbb{Z}^d)} \otimes \cdots \otimes \mathbf{I}_{\ell^2(\mathbb{Z}^d)}}_{j-1 \text{ times}} \otimes \Delta^{(j)} \otimes \underbrace{\mathbf{I}_{\ell^2(\mathbb{Z}^d)} \otimes \cdots \otimes \mathbf{I}_{\ell^2(\mathbb{Z}^d)}}_{n-j \text{ times}},$$

where $\Delta^{(j)}$ denotes the lattice nearest-neighbor Laplacian acting on the jth factor $\mathbb{Z}^d$ of the decomposition $(\mathbb{Z}^d)^n = \mathbb{Z}^d \times \cdots \times \mathbb{Z}^d$, and $\mathbf{I}_{\ell^2(\mathbb{Z}^d)}$ is the identity operator on $\ell^2(\mathbb{Z}^d)$.

2.2. **Basic assumptions.**

2.2.1. *Assumption on* $\mathbf{\Delta}$.

(K) Kinetic energy. $\mathbf{\Delta}$ is the nd-dimensional lattice nearest-neighbor Laplacian:

$$(\mathbf{\Delta}\mathbf{\Psi})(\mathbf{x}) = \sum_{\substack{\mathbf{y}\in\mathbb{Z}^{nd} \\ |\mathbf{y}-\mathbf{x}|_1=1}} (\mathbf{\Psi}(\mathbf{y}) - \mathbf{\Psi}(\mathbf{x})) = \sum_{\substack{\mathbf{y}\in\mathbb{Z}^{nd} \\ |\mathbf{y}-\mathbf{x}|_1=1}} \mathbf{\Psi}(\mathbf{y}) - 2dn\mathbf{\Psi}(\mathbf{x}), \tag{2.1}$$

acting on $\ell^2(\mathbb{Z}^{nd})$, $\mathbf{\Psi} \in \ell^2(\mathbb{Z}^{nd})$, $\mathbf{x} \in \mathbb{Z}^{nd}$.

2.2.2. *Assumptions on* $\mathbf{U}$.

(I) Short-range interaction. The potential energy of inter-particle interaction

$$\mathbf{U}\colon (\mathbb{Z}^d)^n \to \mathbb{R}$$

is of the form

$$\mathbf{U}(\mathbf{x}) = \sum_{1\le i<j\le n} \Phi(x_i, x_j), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^{nd} \tag{2.2}$$

where $\Phi\colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is a symmetric bounded nonnegative function which satisfies

$$\begin{aligned} &\text{there exists } r_0 \in [0,\infty) \text{ such that} \\ &|x_1 - x_2| > r_0 \implies \Phi(x_1, x_2) = 0. \end{aligned} \tag{2.3}$$

We will call r_0 the "range" of interaction $\mathbf{U}$.

Some auxiliary results, concerning eigenvalue concentration (Wegner-type estimates), do not require the interaction $\mathbf{U}$ to have short range. Hence, it is convenient to formulate the following, weaker assumption on $\mathbf{U}$.

(I0) Bounded interaction. The potential energy of inter-particle interaction

$$\mathbf{U}\colon (\mathbb{Z}^d)^n \to \mathbb{R}$$

is of the form (2.2), invariant by permutations of the particles $x_1, \dots, x_n$, and bounded.

2.2.3. *Assumptions on* V. To describe our assumptions on the random potential

$$V\colon \mathbb{Z}^d \times \Omega \to \mathbb{R},$$

we need some notations. Let $Q \subset \mathbb{Z}^d$ be a bounded parallelepiped. We denote by $\xi_Q(\omega)$ the sample mean of the random field V over Q:

$$\xi_Q(\omega) = \langle V\rangle_Q = \frac{1}{|Q|}\sum_{x\in Q} V(x,\omega),$$

and by $\eta_x(\omega)$ the fluctuations relative to the sample mean

$$\eta_x(\omega) = V(x,\omega) - \xi_Q(\omega), \quad x \in Q.$$

Let F_V be the common probability distribution function of the random potential V:

$$F_V(E) = \mathbb{P}\{V(0,\omega) \le E\}.$$

Now denote by $\mathfrak{F}_{V,Q}$ the sigma algebra generated by $\{\eta_x, x \in Q\}$ and by $F_\xi(\,\cdot\,|\mathfrak{F}_{V,Q})$ the conditional distribution function of ξ_Q given $\mathfrak{F}_{V,Q}$:

$$F_\xi(s|\mathfrak{F}_{V,Q}) := \mathbb{P}\left\{\xi_Q \le s|\mathfrak{F}_{V,Q}\right\}.$$

Our assumption on V is as follows:

(P1) External (random) potential energy. The random field $V\colon \mathbb{Z}^d \times \Omega \to \mathbb{R}$ is i.i.d., nonnegative, and for any $R \ge 0$ there exists a function $\nu_R\colon \mathbb{R}_+ \to \mathbb{R}_+$ with the following two properties:

(i) for some $b > 0$, $0 < t \le 1 \implies \nu_R(t) \le C' R^{C''} t^b$,

(ii) **(CM(ν))** For all $Q \subset \mathbb{Z}^d$ with $\operatorname{diam}(Q) \le R$, the conditional distribution function $F_\xi(\,\cdot\,|\mathfrak{F}_{V,Q})$ satisfies, for all $t, s \in \mathbb{R}$:

$$\operatorname{ess\,sup} |F_\xi(t|\mathfrak{F}_{V,Q}) - F_\xi(s|\mathfrak{F}_{V,Q})| \le \nu_R(|t-s|). \tag{2.4}$$

We also assumed that for all $\epsilon > 0$ we have $\mathbb{P}\left\{V(0,\omega) < \epsilon\right\} > 0$, i.e., $\inf \operatorname{supp} F_V = 0$.

In Section 5, this assumption (**P1**) will be replaced by the following one:

(P2) External (random) potential energy. The random field $V\colon \mathbb{Z}^d \times \Omega \to \mathbb{R}$ is i.i.d., and the probability distribution function F_V satisfies

$$\sup_{a\in\mathbb{R}}(F_V(a+\epsilon) - F_V(a)) \le \frac{C}{|\ln\epsilon|^{2A}} \tag{2.5}$$

for $C \in (0,\infty)$ and $A > 24Nd$.

Note that this last condition depends on N.

Remark 1. In the course of the scale induction, the bound (2.5) will be used in a situation where $\epsilon = \mathrm{e}^{-L^\beta}$, $\beta = 1/2$ and $L > 0$ is a large integer. With such a value of ϵ, (2.5) takes the form

$$\sup_{a\in\mathbb{R}}(F_V(a+\mathrm{e}^{-L^\beta}) - F_V(a)) \le C\,L^{-A}. \tag{2.6}$$

Such a power-law estimate is sufficient for the purposes of the MSA: see (3.4)–(3.5). Assumption (**P1**) will give rise to much stronger bounds of the form $\mathrm{e}^{-L^{1/2}+\mathrm{O}(\ln L)}$.

2.3. **The structure of the induction on the number of particles.** Following the general approach of [1,6], we prove localization for systems with a number of particles bounded by some integer $N < \infty$. The value of $N \ge 2$ may be arbitrary, but once it is chosen, it is fixed for the rest of the scaling analysis, and several important parameters used in our scheme depend upon N.

As a result, we carry out a finite induction on the number of particles n varying from 1 (where the well-known results of the 1-particle localization theory can be used) to N. In some notations we emphasize the N-dependence of parameters, while in other instances this dependence is omitted for simplicity.

The inductive step from $n-1$ to n requires information on n'-particle systems for all $1 \le n' \le n-1$. This can be briefly explained by the fact that the induction step involves the analysis of the decompositions of an n-particle system into two subsystems with n' and $n'' = n - n'$ particles, for any $1 \le n' \le n-1$. So it is assumed that all necessary bounds are available for n'-particle systems with any $1 \le n' \le n-1$.

Since our main method is (a multi-particle adaptation of) the MSA, the induction step from $n-1$ to n involves also a scale induction, i.e., induction over a sequence of scales L_k, $k \geq 0$. The starting point of the scale induction requires key bounds of the MSA to be established at the initial scale L_0.

2.4. **Multiscale analysis.** According to the general structure of the MSA, we work with lattice *cubes*

$$\mathbf{C}_L^{(n)}(\mathbf{u}) = \left\{ \mathbf{x} \in \mathbb{Z}^{nd} : |\mathbf{x} - \mathbf{u}| \leq L \right\}. \tag{2.7}$$

The superscript, indicating the number n of particles, will be sometimes omitted, e.g., for $n = 1$. Define its *internal boundary*

$$\partial^- \mathbf{C}_L^{(n)}(\mathbf{u}) = \left\{ \mathbf{v} \in \mathbb{Z}^{nd} : \operatorname{dist}\left(\mathbf{v}, \mathbb{Z}^{nd} \setminus \mathbf{C}_L^{(n)}(\mathbf{u}) \right) = 1 \right\}, \tag{2.8}$$

its *external boundary*

$$\partial^+ \mathbf{C}_L^{(n)}(\mathbf{u}) = \left\{ \mathbf{v} \in \mathbb{Z}^{nd} \setminus \mathbf{C}_L^{(n)}(\mathbf{u}) : \operatorname{dist}\left(\mathbf{v}, \mathbf{C}_L^{(n)}(\mathbf{u}) \right) = 1 \right\}. \tag{2.9}$$

The *cardinality* of a cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ is $|\mathbf{C}_L^{(n)}(\mathbf{u})| = (2L+1)^{nd}$. For simplicity, we will often use the simpler estimate $|\mathbf{C}_L^{(n)}(\mathbf{u})| \leq (3L)^{nd}$. Let an n-particle cube

$$\mathbf{C}_L^{(n)}(\mathbf{u}) = C_L(u_1) \times \cdots \times C_L(u_n) \subset \mathbb{Z}^{nd}.$$

We define its *projection* by

$$\varPi \mathbf{C}_L^{(n)}(\mathbf{u}) = C_L(u_1) \cup \cdots \cup C_L(u_n) \subset \mathbb{Z}^d. \tag{2.10}$$

More generally, for any nonempty subset $\mathcal{J} \subset \{1, \ldots, n\}$ we define its $\mathcal{J}$-projection by

$$\varPi_{\mathcal{J}} \mathbf{C}_L^{(n)}(\mathbf{u}) = \bigcup_{j \in \mathcal{J}} C_L(u_j) \subset \mathbb{Z}^d. \tag{2.11}$$

We define the restriction of the Hamiltonian $\mathbf{H}^{(n)}$ to a cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ by

$$\mathbf{H}^{(n)}_{\mathbf{C}_L^{(n)}(\mathbf{u})} = \mathbf{H}^{(n)}\big|_{\mathbf{C}_L^{(n)}(\mathbf{u})}$$

with Dirichlet boundary conditions on $\partial^+ \mathbf{C}_L^{(n)}(\mathbf{u})$.

We denote its spectrum by $\sigma\big(\mathbf{H}^{(n)}_{\mathbf{C}_L^{(n)}(\mathbf{u})}\big)$ and its resolvent by

$$\mathbf{G}_{\mathbf{C}_L^{(n)}(\mathbf{u})}(E) := \left(\mathbf{H}^{(n)}_{\mathbf{C}_L^{(n)}(\mathbf{u})} - E \right)^{-1}, \quad E \in \mathbb{R} \setminus \sigma\left(\mathbf{H}^{(n)}_{\mathbf{C}_L^{(n)}(\mathbf{u})} \right). \tag{2.12}$$

Its matrix elements $\mathbf{G}_{\mathbf{C}_L^{(n)}(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E)$ are usually called the *Green functions* of the operator $\mathbf{H}^{(n)}_{\mathbf{C}_L^{(n)}(\mathbf{u})}$.

A permutation $\tau \in \mathfrak{S}_n$ acts on a configuration $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^{nd}$ of n particles by $\tau \mathbf{x} = (x_{\tau^{-1}(1)}, \ldots, x_{\tau^{-1}(n)})$. We define the *symmetrized distance* between two configurations $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{nd}$ by

$$d_{\mathrm{S}}(\mathbf{x}, \mathbf{y}) = \min_{\tau \in \mathfrak{S}_n} |\mathbf{x} - \tau \mathbf{y}|. \tag{2.13}$$

Definition 1 (ℓ-distant)**.** Let $N \geq 1$ be fixed. Two subsets $\mathbf{A}$, $\mathbf{B} \subset \mathbb{Z}^{nd}$, $1 \leq n \leq N$ are called ℓ-distant if

$$d_{\mathrm{S}}(\mathbf{A}, \mathbf{B}) > 11 N \ell.$$

The multiscale analysis is based on a length scale $\{L_k\}_{k \geq 0}$ which is chosen as follows.

Definition 2 (length scale)**.** The length-scale $\{L_k\}_{k\geq 0}$ is a sequence of integers defined by the initial length-scale $L_0 > 2$, and by the recurrence relation

$$L_{k+1} = \lfloor L_k^\alpha \rfloor,$$

where $1 < \alpha < 2$ is some fixed number. In this paper, $\alpha = 3/2$.

This length scale $\{L_k\}_{k\geq 0}$ is assumed to be chosen at the beginning of the multiscale analysis, except that in the course of the analysis it is often required that L_0 be large enough.

Definition 3 (E-resonant)**.** Let $n \geq 1$, $\beta = 1/2$, and $E \in \mathbb{R}$ be given. A cube $\mathbf{C}_L^{(n)}(\mathbf{v}) \subset \mathbb{Z}^{nd}$ of size $L \geq 2$ is called E-resonant (E-R) if

$$\operatorname{dist}\Big[E, \sigma\big(\mathbf{H}^{(n)}_{\mathbf{C}_L^{(n)}(\mathbf{v})}\big)\Big] < \mathrm{e}^{-L^\beta}. \tag{2.14}$$

Otherwise it is called E-non-resonant (E-NR).

The next definition depends on the parameter $\alpha > 1$ which governs the length scale of our multiscale analysis.

Definition 4 (E-completely non-resonant)**.** Let $E \in \mathbb{R}$ be given. A cube $\mathbf{C}_L^{(n)}(\mathbf{v}) \subset \mathbb{Z}^{nd}$ of size $L \geq 2$ is called E-completely non-resonant (E-CNR) if it does not contain any E-R cube of size $\geq L^{1/\alpha}$. In particular, $\mathbf{C}_L(\mathbf{v})$ is itself E-NR.

Definition 5 ((E, m)-singular)**.** Let $m > 0$ and $E \in \mathbb{R}$ be given. Let $N \geq 1$ be fixed. A cube $\mathbf{C}_L^{(n)}(\mathbf{u}) \subset \mathbb{Z}^{nd}$, $1 \leq n \leq N$ is called (E, m)-nonsingular ((E, m)-NS) if

$$\max_{\mathbf{v}\in\partial^-\mathbf{C}_L^{(n)}(\mathbf{u})} \Big|\mathbf{G}_{\mathbf{C}_L^{(n)}(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E)\Big| \leq \mathrm{e}^{-\gamma(m,L,n)L}, \tag{2.15}$$

where $\gamma(m, L, n) = \gamma(m, L, n, N)$ is defined by

$$\gamma(m, L, n) = m(1 + L^{-1/8})^{N-n+1} > m. \tag{2.16}$$

Otherwise it is called (E, m)-singular ((E, m)-S).

2.5. **Multi-particle localization at low energies.** We prove our main result on multi-particle spectral localization under two different sets of assumptions:

- (**K**), (**I**), and (**P1**),
- (**K**), (**I**), and (**P2**), which is less restrictive than (**P1**).

As we have already pointed out, assumption (**P1**), introduced in [4], leads to a simpler proof (compared to the scheme described in [6]) and to more optimal decay bounds for eigenfunctions, especially in finite cubes. The latter improvement does not manifest itself in the qualitative statement on pure point spectrum and *asymptotic* exponential decay of eigenfunctions. We discuss this issue in more detail in Section 5.

Theorem 1 (multi-particle localization at low energies)**.** *Let $\mathbf{H}^{(N)}(\omega)$ be a random N-particle Hamiltonian of the form*

$$\mathbf{H}^{(N)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^{N} V(x_j, \omega) + \mathbf{U},$$

where $\mathbf{\Delta}$, $\mathbf{U}$ satisfy (**K**), (**I**), *respectively and let*

$$E^0 = \inf \sigma(\mathbf{H}(\omega)) \quad \mathbb{P}\text{-}a.s. \tag{2.17}$$

(A) *If the random potential V satisfies* (**P1**)*, or if it satisfies* (**P2**) *and is bounded from below, then there exists $E^* > E^0$ such that with $\mathbb{P}$-probability one,*

(i) *the spectrum of $\mathbf{H}(\omega)$ in $(E^0, E^*]$ is nonempty and pure point,*
(ii) *any eigenfunction $\mathbf{\Psi}_i(\mathbf{x}, \omega)$ with eigenvalue $E_i(\omega) \in [E^0, E^*]$ is exponentially decaying at infinity: there exist a non-random constant $m > 0$ and a random constant $C_i(\omega) > 0$ such that*

$$|\mathbf{\Psi}_i(\mathbf{x}, \omega)| \le C_i(\omega) \mathrm{e}^{-m|\mathbf{x}|}. \tag{2.18}$$

(B) *If the random potential V satisfies* (**P2**) *and is unbounded from below:*

$$\inf\{E \in \mathbb{R} : F_V(E) > 0\} = -\infty,$$

then, for any positive m, there exists $E^*(m) \in \mathbb{R}$ such that (i) *and* (ii) *are valid for energies $E \le E^*(m)$ with the given m as rate of decay.*

Proof. See Sections 3–4 for (A) under (**P1**), and Section 5 for (A) under (**P2**) and for (B). □

2.6. **Multi-particle localization under weak interaction.** To formulate our result on stability of multi-particle localization under perturbations by weak interactions, it is convenient to write the N-particle Hamiltonian as follows:

$$\mathbf{H}^{(N)}(\omega) = \mathbf{H}^{(N)}_{g,h}(\omega) = -\mathbf{\Delta} + g\mathbf{V}(\omega) + h\mathbf{U}, \quad g, h \in \mathbb{R}, \tag{2.19}$$

where $\mathbf{V}(\omega) = \mathbf{V}(\mathbf{x}, \omega) = V(x_1, \omega) + \cdots + V(x_N, \omega)$ and h is a parameter measuring the strength of the non-random interaction $\mathbf{U}$ satisfying (**I**) (boundedness and short range). For $h = 0$, one has a system of N non-interacting quantum particles subject to a common random potential $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$.

Technically, we prove in this paper stability of the MPMSA bounds obtained in [6], for the principal – and most interesting – application is to one-dimensional multi-particle systems with arbitrarily small disorder amplitude $|g| > 0$. As it is well-known, Anderson localization occurs in one-dimensional lattice systems with any non-constant i.i.d. random potential: no regularity of its marginal probability distribution is required. In particular, the i.i.d. potential may even take a finite number $K \ge 2$ of values. So, we prefer here a more general assumption (**P2**) (log-Hölder continuity of the marginal distribution). In this connection, we would like to emphasize that more singular potentials (e.g., those taking a finite number of values) cannot be treated by existing methods (neither MPMSA nor MPFMM), since these methods treat N-particle Hamiltonians with $N > 1$ as Nd-dimensional lattice operators, and specifically one-dimensional analytic techniques (products of random matrices and Furstenberg theory) fail to apply to such models.

Theorem 2 (multi-particle localization under weak interaction)**.** *Let $g, h \in \mathbb{R}$ and let*

$$\mathbf{H}^{(N)}_{g,h}(\omega) = -\mathbf{\Delta} + g\mathbf{V}(\omega) + h\mathbf{U}$$

be an N-particle hamiltonian in $\ell^2(\mathbb{Z}^{Nd})$, with $\mathbf{\Delta}$, $\mathbf{U}$, and V satisfying (**K**)*,* (**I**)*, and* (**P2**)*, respectively.*

We assume that $\mathbf{H}^{(N)}_{g,0}(\omega)$, i.e.*, the Hamiltonian without inter-particle interaction, satisfies the following condition: there exists a nonempty energy interval $I \subset \mathbb{R}$ such*

that, for all $E \in I$,

$$\mathbb{P}\left\{ \mathbf{C}^{(N)}_{L_0}(\mathbf{u}) \text{ is } (E,m)\text{-}S \right\} \leq L_0^{-2p}, \tag{2.20}$$
$$\text{for } p > 6Nd, \quad m > CL_0^{-1/2},$$

and with L_0 large enough.

Then there exists $h^* > 0$ such that for any $h \in (-h^*, h^*)$ the Hamiltonian $\mathbf{H}^{(N)}_{g,h}(\omega)$, with interaction of amplitude $|h|$, exhibits Anderson localization in the interval I, i.e., *with $\mathbb{P}$-probability one,*

(i) *the spectrum of $\mathbf{H}^{(N)}_{g,h}(\omega)$ in I is pure point,*
(ii) *the eigenfunctions $\mathbf{\Psi}_i(\mathbf{x},\omega)$ relative to eigenvalues $E_i(\omega) \in I$ are exponentially decaying at infinity:*

$$|\mathbf{\Psi}_i(\mathbf{x},\omega)| \leq C_i(\omega)\mathrm{e}^{-m|\mathbf{x}|}, \quad m > 0.$$

Proof. See Section 6.1. □

The following statement relative to N-particle systems in $\mathbb{Z}$ will be derived from Theorem 2 since, for $d = 1$, we are able to establish the estimate (2.20).

Corollary 1 (N-particle localization under weak interaction, $d = 1$)**.** *Let*

$$\mathbf{H}^{(N)}_{g,h}(\omega) = -\mathbf{\Delta} + g\mathbf{V}(\mathbf{x},\omega) + h\mathbf{U}, \quad g, h \in \mathbb{R}$$

be a random operator in $\ell^2(\mathbb{Z}^N)$ describing an N-particle quantum system in the one-dimensional lattice $\mathbb{Z}$ subject to the external random potential $g\mathbf{V}(\omega)$ and to the inter-particle interaction $h\mathbf{U}$, with $\mathbf{\Delta}$, $\mathbf{U}$, and $V\colon \mathbb{Z}\times\Omega \to \mathbb{R}$ satisfying (**K**)*,* (**I**)*, and* (**P2**)*, respectively.*

Then for any $g \neq 0$ there exist $h^* > 0$ and a nonempty interval $I \subset \mathbb{R}$ such that $\mathbf{H}^{(N)}_{g,h}(\omega)$ exhibits complete Anderson localization in I for all $h \in (-h^*, h^*)$, i.e., *with $\mathbb{P}$-probability one,*

(i) *the spectrum of $\mathbf{H}^{(N)}_{g,h}(\omega)$ in I is pure point,*
(ii) *there is an orthogonal basis $\{\psi_i(x,\omega)\}_{i\in\mathbb{N}}$ in $\ell^2(\mathbb{Z}^N)$ consisting of eigenfunctions $\psi_i(x,\omega)$ relative to eigenvalues $E_i(\omega) \in I$ with exponential decay at infinity:*

$$|\psi_i(x,\omega)| \leq C_i(\omega)\mathrm{e}^{-m|x|}, \quad m > 0.$$

Proof. See Section 6.2. □

Taking into account that our approach is perturbative, it is clear that the value of the "mass" $m = m(g, h, F_V) > 0$, depending upon the PDF F_V of the random potential $V(\omega)$, for $|h|$ small enough is close to the upper Lyapunov exponent of the corresponding single-particle one-dimensional Anderson model. For bounded potentials and small $|g|$, the latter is known to be of order of $\mathrm{O}(g^2)$ for energies E away from the center and edges of the interval $[0, 4]$ giving the spectrum of the nonnegative lattice Laplacian $-\Delta$ in $\mathbb{Z}$ (cf. [7, 13, 14] and references therein).

3. The N-particle MSA scheme under (K), (I) and (P1)

3.1. **The general structure of the MPMSA.** Recall again that we always assume $N \geq 2$ to be chosen and fixed. We consider only systems with no more than N particles. The proof is by induction on two parameters, the number of particles $n = 1, \ldots, N$ and the scale index $k \geq 0$. See section 2.3.

Assume that $L_0 \geq 8r_0$. Given $1 \leq n \leq N$, the following property for some $E^* > E^0$ will play an important role in our main strategy.

($\mathbf{DS}.k,n,N$). For any pair of L_k-distant cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ and $\mathbf{C}^{(n)}_{L_k}(\mathbf{v})$

$$\mathbb{P}\left\{\exists\, E \in I : \mathbf{C}^{(n)}_{L_k}(\mathbf{u}),\ \mathbf{C}^{(n)}_{L_k}(\mathbf{v}) \text{ are } (E,m)\text{-S}\right\} \leq L_k^{-2p\,4^{N-n}} \tag{3.1}$$

where $m > 0$, $p > 6Nd$, and $I = [E^0, E^*]$ with $E^* > E^0$, are fixed.

By Definition 1, if for N fixed and $1 \leq n \leq N$ two cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ and $\mathbf{C}^{(n)}_{L_k}(\mathbf{v})$ are L_k-distant then $|\mathbf{u} - \mathbf{v}| > 11\,NL_k$. The role of this lower bound on the distance between their centers will become clear later. We denote $(\mathbf{DS}.k,N) := (\mathbf{DS}.k,N,N)$.

($\mathbf{DS}.k,N$). For any pair of L_k-distant cubes $\mathbf{C}^{(N)}_{L_k}(\mathbf{u})$ and $\mathbf{C}^{(N)}_{L_k}(\mathbf{v})$

$$\mathbb{P}\left\{\exists\, E \in I : \mathbf{C}^{(N)}_{L_k}(\mathbf{u}),\ \mathbf{C}^{(N)}_{L_k}(\mathbf{v}) \text{ are } (E,m)\text{-S}\right\} \leq L_k^{-2p} \tag{3.2}$$

where $m > 0$, $p > 6Nd$, and $I = [E^0, E^*]$ with $E^* > E^0$, are fixed.

Theorem 3. *Let $\mathbf{H}^{(N)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^N V(x_j,\omega) + \mathbf{U}$, where $\mathbf{\Delta}$, $\mathbf{U}$, V satisfy* (**K**), (**I**), (**P1**), *respectively.*

Then for any $p > 0$ there exist $E^* = E^*(p,N) > E^0$ and $m > 0$ such that ($\mathbf{DS}.k,N$) *holds true for all $k \geq 0$ provided L_0 is large enough.*

Proof. The proof occupies Sections 3 and 4. ☐

Theorem 4. *Let $\mathbf{H}^{(N)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^N V(x_j,\omega) + \mathbf{U}$, where $\mathbf{\Delta}$, $\mathbf{U}$, V satisfy* (**K**), (**I**), (**P1**), *respectively. Suppose* ($\mathbf{DS}.k,N$) *holds for some $E^* > E^0$ and all $k \geq 0$.*

Then with $\mathbb{P}$-probability one,

(i) *the spectrum of $\mathbf{H}^{(N)}(\omega)$ in $[E^0, E^*]$ is pure point,*
(ii) *there exists a non-random number $m > 0$ such that all eigenfunctions $\mathbf{\Psi}_i(\mathbf{x},\omega)$ of $\mathbf{H}^{(N)}(\omega)$ with eigenvalues $E_i(\omega) \in [E^0, E^*]$ are exponentially decaying at infinity with rate m:*

$$|\mathbf{\Psi}_i(\mathbf{x},\omega)| \leq C_i(\omega)\mathrm{e}^{-m|\mathbf{x}|}. \tag{3.3}$$

Proof. The idea of this theorem goes back to the pioneering works [8,10] and it can be proven essentially in the same way as in these papers, with minor adaptations. See, e.g., [6]. ☐

It is readily seen that Theorem 4 combined with Theorem 3 implies Theorem 1 (A) if V satisfies (**P1**). It remains, therefore, to prove Theorem 3.

3.2. **Eigenvalue concentration bounds.** Given $n \geq 1$ and $L_0 > 2$, we introduce the following two properties ($\mathbf{WS1}.n$) and ($\mathbf{WS2}.n$) of random Hamiltonians $\mathbf{H}_{\mathbf{C}^{(n)}_L(\mathbf{v})}$.

($\mathbf{WS1}.n$). For any $L \geq L_0$, any cube $\mathbf{C}^{(n)}_L(\mathbf{x})$, and any $E \in \mathbb{R}$,

$$\mathbb{P}\big\{\mathbf{C}^{(n)}_L(\mathbf{x}) \text{ is not } E\text{-CNR}\big\} < L^{-q}. \tag{3.4}$$

($\mathbf{WS2}.n$). For any $L \geq L_0$ and any pair of L-distant cubes $\mathbf{C}^{(n)}_L(\mathbf{x})$ and $\mathbf{C}^{(n)}_L(\mathbf{y})$,

$$\mathbb{P}\left\{\exists\, E \in \mathbb{R},\ \text{neither } \mathbf{C}^{(n)}_L(\mathbf{x}) \text{ nor } \mathbf{C}^{(n)}_L(\mathbf{y}) \text{ is } E\text{-CNR}\right\} < L^{-q} \tag{3.5}$$

for some $q = q(n)$ large enough.

More precisely, for a fixed number of particles N, and any $1 \le n \le N$, we always assume that the parameter $p > 0$ in $(\mathbf{DS}.k, n, N)$ or $(\mathbf{DS}.k, N)$ and the exponent q in $(\mathbf{WS1}.n)$ or $(\mathbf{WS2}.n)$ satisfy

$$p > 6Nd, \quad q > 4^N p > 6 \times 4^N Nd, \tag{3.6}$$

i.e., in terms of the parameter α (we set $\alpha = 3/2$):

$$p > 4Nd\alpha, \quad q > 4^N p > 4^{N+1} Nd\alpha.$$

Lemma 1 ([4])**.** *Under assumptions* $(\mathbf{K})$, $(\mathbf{I0})$, *and* $(\mathbf{P1})$ *or* $(\mathbf{P2})$, *both properties* $(\mathbf{WS1}.n)$ *and* $(\mathbf{WS2}.n)$ *hold true for any positive integer* n *provided* L_0 *is large enough.*

Lemma 2. *Under assumptions* $(\mathbf{K})$, $(\mathbf{I0})$, *and* $(\mathbf{P2})$, *for any cube* $\mathbf{C}^{(n)}_L(\mathbf{u})$, $L \ge 1$, $1 \le n \le N$, *and any* $s \in (0, 1]$

$$\mathbb{P}\left\{ \left\| \mathbf{G}_{\mathbf{C}^{(n)}_L(\mathbf{u})}(E) \right\| \ge s^{-1} \right\} \le C \left|\mathbf{C}_L(\mathbf{u})\right|^2 \times \left|\ln s\right|^{-A} \tag{3.7}$$

where $A > 24Nd$ *is as* (2.5), *which is part of assumption* $(\mathbf{P2})$.

3.3. Initial scale estimates.

Let $N > 1$ be fixed and let $1 \le n \le N$. We observe that $(\mathbf{DS}.0, n, N)$ follows from the following stronger property:

$(\mathbf{S}.0, n, N)$**.** For any $\mathbf{x} \in \mathbb{Z}^{nd}$,

$$\mathbb{P}\left\{ \exists\, E \in [E^0, E^*] \text{ such that } \mathbf{C}^{(n)}_{L_0}(\mathbf{x}) \text{ is } (E, m)\text{-S} \right\} \le L_0^{-2p\, 4^{N-n}} \tag{3.8}$$

where $m > 0$, $p > 6Nd$, and $I = [E^0, E^*]$ with $E^* > E^0$, are fixed.

We denote $(\mathbf{S}.0, N) := (\mathbf{S}.0, N, N)$.

$(\mathbf{S}.0, N)$**.** For any $\mathbf{x} \in \mathbb{Z}^{Nd}$,

$$\mathbb{P}\left\{ \exists\, E \in [E^0, E^*] \text{ such that } \mathbf{C}^{(N)}_{L_0}(\mathbf{x}) \text{ is } (E, m)\text{-S} \right\} \le L_0^{-2p} \tag{3.9}$$

where $m > 0$, $p > 6Nd$, and $I = [E^0, E^*]$ with $E^* > E^0$, are fixed.

Indeed, $(\mathbf{DS}.0, n, N)$ assesses the probability to have *two* cubes to be singular at the same energy. Such an event implies that the first cube (as well as the second one) is singular, and for one cube one can apply $(\mathbf{S}.0, n, N)$. So, for our purposes, it suffices to establish $(\mathbf{S}.0, n, N)$ for any $1 \le n \le N$.

The initial scale bound (3.8) — hence also $(\mathbf{DS}.0, n, N)$ — will be proven with the help of a similar property of single-particle operators. Specifically, we need the following two results from the single-particle localization theory.

Lemma 3 (Combes–Thomas estimate)**.** *Consider a Schrödinger operator*

$$H = -\Delta + W(x)$$

acting in $\ell^2(\Lambda)$, $\Lambda \subset \mathbb{Z}^\nu$, $\nu \ge 1$, *with an arbitrary*[1] *potential* $W\colon \Lambda \to \mathbb{R}$. *Suppose that*

- $E \in \mathbb{R}$ *is such that* $\operatorname{dist}(E, \sigma(H)) \ge \eta$ *with* $0 < \eta \le 1$.

Then for all $x, y \in \Lambda$

$$\left| (H - E)^{-1}(x, y) \right| \le \frac{2}{\eta}\, \mathrm{e}^{-\frac{\eta}{12\nu}|x-y|_1} \le \frac{2}{\eta}\, \mathrm{e}^{-\frac{\eta}{12\nu}|x-y|}. \tag{3.10}$$

Proof. See e.g. [11, Theorem 11.2] □

[1] This includes the cases of single- and multi-particle operators, that differ only by their potentials.

Lemma 4. *Let $H^{(1)}(\omega) = -\Delta + V(x,\omega)$ be a random single-particle lattice Schrödinger operator in $\ell^2(\mathbb{Z}^d)$. The random variables $V(x,\omega)$ are i.i.d., non constant and nonnegative.*

Then, for any $C > 0$ and arbitrary large $L_0 > 0$ there exists $c > 0$ such that the lowest eigenvalue $E_0^{(1)}(\omega)$ of the restriction $H^{(1)}_{C_{L_0}(u)}(\omega)$ of $H^{(1)}(\omega)$ to a cube $C_{L_0}(u)$ with Dirichlet[2] *boundary conditions satisfies*

$$\mathbb{P}\big\{E_0^{(1)}(\omega) \le 2CL_0^{-1/2}\big\} \le \mathrm{e}^{-c|C_{L_0}(u)|^{1/4}}. \tag{3.11}$$

Proof. See [11, Theorem 11.4] and its proof. Lemma 4 follows actually from the study of Lifshitz tails, see [11, Lemma 6.4] and [15, Theorem 2.1.3]. □

In our earlier work [9], we have already inferred the initial scale MPMSA estimate for the two-particle lattice Anderson model from the results given by Lemma 3 and Lemma 4. The adaptation to an arbitrary number of particles n is straightforward.

Lemma 5. *Let $\mathbf{H}^{(n)}(\omega) = -\mathbf{\Delta} + V(x_1,\omega) + \cdots + V(x_n,\omega) + \mathbf{U}(\mathbf{x})$ be an n-particle random Schrödinger operator in $\ell^2(\mathbb{Z}^{nd})$. Suppose*

(i) *the interaction $\mathbf{U}$ is bounded and nonnegative*[3],

(ii) *the random external potential $V(x,\omega)$ is i.i.d. and nonnegative.*

Then, for any $C > 0$ and arbitrary large $L_0 > 0$ there exists $c > 0$ such that the lowest eigenvalue $E_0^{(n)}(\omega)$ of the restriction $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega)$ of $\mathbf{H}^{(n)}(\omega)$ to a cube $\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \subset \mathbb{Z}^{nd}$ satisfies

$$\mathbb{P}\big\{E_0^{(n)}(\omega) \le 2CL_0^{-1/2}\big\} \le \mathrm{e}^{-c|C_{L_0}(u)|^{1/4}}. \tag{3.12}$$

Proof. Since the interaction potential $\mathbf{U}$ is nonnegative, then by the min-max principle the lowest eigenvalue $E_0^{(n)}(\omega)$ of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}}$ is bounded from below by the lowest eigenvalue $\widetilde{E}_0^{(n)}(\omega)$ of the operator

$$\widetilde{\mathbf{H}}^{(n)}(\omega) = -\mathbf{\Delta} + V(x_1,\omega) + \cdots + V(x_n,\omega).$$

The latter operator can be rewritten as follows:

$$\widetilde{\mathbf{H}}^{(n)}(\omega) = \sum_{j=1}^{n} \underbrace{\mathbf{1}_{\ell^2(\mathbb{Z}^d)} \otimes \cdots \otimes \mathbf{1}_{\ell^2(\mathbb{Z}^d)}}_{j-1 \text{ times}} \otimes H_j^{(1)} \otimes \underbrace{\mathbf{1}_{\ell^2(\mathbb{Z}^d)} \otimes \cdots \otimes \mathbf{1}_{\ell^2(\mathbb{Z}^d)}}_{n-j \text{ times}}$$

where $H_j^{(1)} = -\Delta^{(j)} + V(x_j,\omega)$, $j = 1,\dots,n$. Hence the lowest eigenvalue $\widetilde{E}_0^{(n)}(\omega)$ of $\widetilde{\mathbf{H}}^{(n)}(\omega)$ has the following form:

$$\widetilde{E}_0^{(n)}(\omega) = \sum_{j=1}^{n} E_{0,j}^{(1)}(\omega)$$

where $E_{0,j}^{(1)}$ is the lowest eigenvalue of $H_j^{(1)}$. All eigenvalues $E_{0,j}^{(1)}$ are nonnegative due to the nonnegativity of the operators $H_j^{(1)}$ with nonnegative external potential, so that for any $s \ge 0$,

$$\mathbb{P}\left\{\widetilde{E}_0^{(n)}(\omega) \le s\right\} \le \mathbb{P}\left\{E_{0,1}^{(1)}(\omega) \le s\right\}.$$

[2]In the proof, one usually considers first an operator with Neumann boundary conditions, and then applies Dirichlet–Neumann bracketing.

[3]The assumption that $\mathbf{U}$ has short range, or is decaying at infinity, is not required.

Therefore, Lemma 5 follows from Lemma 4 applied to the single-particle Schrödinger operator $H_1^{(1)}$. □

Remark 2. For the assertion of Lemma 5 to be useful, i.e. in order to ensure that the spectrum of random operators $\mathbf{H}^{(n)}(\omega)$ in $\ell^2(\mathbb{Z}^{nd})$ is nonempty in any interval $[0,\epsilon]$, $\epsilon > 0$, one would need to require that the marginal probability density function F_V satisfies

$$\mathcal{E}^* := \inf\{E \in \mathbb{R} : F_V(E) > 0\} = 0.$$

Formally, however, the assertion of the lemma remains valid even for $\mathcal{E}^* > 0$. In that case, owing to nonnegativity of the operators $-\Delta^{(j)}$, one has $E_{0,j}^{(1)} \geq \mathcal{E}^* > 0$, so that, trivially,

$$\mathbb{P}\left\{\widetilde{E}_0(\omega) < n\mathcal{E}^*\right\} = 0.$$

Theorem 5 (initial scale estimate). *Let $N > 1$ be fixed and $\mathbf{H}^{(N)}(\omega) = -\boldsymbol{\Delta} + V(x_1,\omega) + \cdots + V(x_N,\omega) + \mathbf{U}$, where $\boldsymbol{\Delta}$, $\mathbf{U}$, V satisfy* (**K**), (**I**), (**P1**), *respectively.*

Then for any $p > 6Nd$, for any L_0 large enough and $0 < C < \infty$, there exist $E^* > E^0$ and $m \geq CL_0^{-1/2} > 0$ such that properties $(\mathbf{S}.0,n,N)$ and $(\mathbf{DS}.0,n,N)$ hold true for any $1 \leq n \leq N$.

Proof. Apply Lemma 3 and Lemma 5. □

Remark 3. Clearly, the inequality $m \geq CL_0^{-1/2}$ implies $m \geq CL_k^{-1/2}$ for all $k \geq 0$. Further, by definition of the function $\gamma(m,L)$ — see (2.16), for all $L > 0$ and $1 \leq n \leq N$, $\gamma(m,L,n) > m$. Therefore, if one proves $(\mathbf{DS}.0,n,N)$ with $m \geq CL_0^{-1/2}$, then the lower bound on the decay exponent of Green functions, required at each subsequent step k of the scale induction, will be guaranteed.

For future use we also give the next statement which treat the case of external random potentials which are bounded from below.

Lemma 6 (V bounded from below). *Let $\mathbf{H}^{(n)}(\omega) = -\boldsymbol{\Delta} + V(x_1,\omega) + \cdots + V(x_n,\omega) + \mathbf{U}(\mathbf{x})$ be an n-particle random Schrödinger operator in $\ell^2(\mathbb{Z}^{nd})$. Suppose*

(i) *the interaction $\mathbf{U}$ is bounded and denote*

$$\mu = \inf_{\mathbf{x}\in\mathbb{Z}^{nd}} \mathbf{U}(\mathbf{x}),$$

(ii) *the random external potential $V(x,\omega)$ is i.i.d., satisfies* (**P2**), *and is bounded from below: there is $\mathcal{E}^* \in \mathbb{R}$ such that the probability distribution function F_V satisfies*

$$\inf\{E \in \mathbb{R} : F_V(E) > 0\} = \mathcal{E}^*.$$

Then, for any $C > 0$ and arbitrary large $L_0 > 0$ there exists $c > 0$ such that the lowest eigenvalue $E_0(\omega)$ of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega)$ satisfies

$$\mathbb{P}\left\{E_0(\omega) \leq n\mathcal{E}^* + \mu + 2CL_0^{-1/2}\right\} \leq \mathrm{e}^{-c|C_{L_0}(u)|^{1/4}}. \tag{3.13}$$

Here, $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega)$ denotes the restriction of $\mathbf{H}^{(n)}(\omega)$ to the cube $\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \subset \mathbb{Z}^{nd}$.

Proof. Let us set

$$\tilde{V}(x,\omega) = V(x,\omega) - \mathcal{E}^*$$
$$\widetilde{\mathbf{U}}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) - \mu.$$

Since $V(x,\omega) \geq \mathcal{E}^*$ almost surely and $\mathbf{U}(\mathbf{x}) \geq \mu$ we have

$$\tilde{V}(x,\omega) \geq 0 \quad \text{and} \quad \widetilde{\mathbf{U}}(\mathbf{x}) \geq 0.$$

Therefore,

$$\begin{aligned}
\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}}(\omega) &= -\boldsymbol{\Delta} + \sum_{j=1}^{n} V(x_j,\omega) + \mathbf{U}(\mathbf{x}) \\
&= \underbrace{-\boldsymbol{\Delta} + \tilde{V}(x_1,\omega) + \cdots + \tilde{V}(x_n,\omega)}_{\widetilde{\mathbf{H}}^{(n)}_{\mathbf{C}^{(n)}_{L_0}}(\omega)} + \widetilde{\mathbf{U}}(\mathbf{x}) + n\mathcal{E}^* + \mu
\end{aligned}$$

so that the lowest eigenvalue $E_0(\omega)$ of the random Hamiltonian $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}}(\omega)$ satisfies

$$E_0(\omega) = \widehat{E}_0(\omega) + n\mathcal{E}^* + \mu,$$

where $\widehat{E}_0(\omega) \geq 0$ is the lowest eigenvalue of $\widetilde{\mathbf{H}}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega) + \widetilde{\mathbf{U}}(\mathbf{x})$. Next

$$E_0(\omega) \leq n\mathcal{E}^* + \mu + 2CL_0^{-1/2} \iff \widehat{E}_0(\omega) \leq 2CL_0^{-1/2}.$$

Thus,

$$\begin{aligned}
\mathbb{P}\left\{E_0(\omega) \leq n\mathcal{E}^* + \mu + 2CL_0^{-1/2}\right\} &\leq \mathbb{P}\left\{\widehat{E}_0(\omega) \leq 2CL_0^{-1/2}\right\} \\
&\leq \mathrm{e}^{-c|C_{L_0}(u)|^{1/4}}.
\end{aligned}$$

Since $\widetilde{\mathbf{U}}(\mathbf{x})$ and $\tilde{V}(x,\omega)$ are nonnegative, Lemma 6 follows by Lemma 5. □

Now we also treat the case where the random potential V is unbounded from below; probably, the most important application is to Gaussian i.i.d. random fields. A particularity of this class of random potentials is that the "mass" $m > 0$ here can be made arbitrarily large, in the energy band $\{E \leq E^* < 0\}$ with sufficiently large $|E^*|$, depending upon the desired value of the decay exponent m. Also, the proof of the initial scale bound in such a case is much simpler: one can use essentially the same argument as in the case of "strong disorder", without appealing to the Lifshitz tails asymptotics.

Lemma 7 (V unbounded from below)**.** *Let* $\mathbf{H}^{(n)}(\omega) = -\boldsymbol{\Delta} + V(x_1,\omega) + \cdots + V(x_n,\omega) + \mathbf{U}(\mathbf{x})$ *be an n-particle random Schrödinger operator in $\ell^2(\mathbb{Z}^{nd})$. Suppose*

(i) *the interaction* $\mathbf{U}$ *is bounded,*
(ii) *the random external potential $V(x,\omega)$ is i.i.d., satisfies assumption* (**P2**)*, and is unbounded from below:*

$$\inf\{E \in \mathbb{R} : F_V(E) > 0\} = -\infty.$$

Let $\eta > 0$, $q > 0$ and $L_0 \geq 2$ be given. Then there exists $E^* = E^*(\eta, q, L_0) < 0$ such that the lowest eigenvalue $E_0^{(n)}(\omega)$ of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega)$ satisfies the estimate

$$\mathbb{P}\left\{E_0^{(n)}(\omega) \leq E^* + \eta\right\} \leq L_0^{-q}. \tag{3.14}$$

Here, $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_0}(\mathbf{u})}(\omega)$ denotes the restriction of $\mathbf{H}^{(n)}(\omega)$ to $\mathbf{C}^{(n)}_{L_0}(\mathbf{u}) \subset \mathbb{Z}^{nd}$.

Proof. First of all, note that, by nonnegativity of the kinetic energy operator and boundedness of the interaction $\mathbf{U}$,

$$-\mathbf{\Delta} + \mathbf{U} \geq -C \cdot \mathbf{1}$$

with $0 \leq C := \|\mathbf{U}\| < \infty$. Therefore, for any $s < 0$ and any fixed $L_0 \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}\left\{E_0^{(n)} < ns\right\} &\leq \mathbb{P}\Big\{\min_{\mathbf{x}\in\mathbf{C}^{(n)}_{L_0}(\mathbf{u})} \mathbf{V}(\mathbf{x},\omega) < ns + C\Big\} \\
&\leq \mathbb{P}\Big\{\min_{x\in\Pi\mathbf{C}^{(n)}_{L_0}(\mathbf{u})} V(x,\omega)) < s + Cn^{-1}\Big\} \\
&\leq |\Pi\mathbf{C}^{(n)}_{L_0}(\mathbf{u})| \times F_V(s + Cn^{-1}) \xrightarrow[s\to-\infty]{} 0
\end{aligned}$$

where $\mathbf{V}(\mathbf{x};\omega) = V(x_1,\omega) + \cdots + V(x_n,\omega)$. Thus for any fixed integer $L_0 \geq 2$, any $\eta > 0$ and $q > 0$ there exists E^* such that (3.14) holds true. ☐

3.4. **The scale induction step.** It remains to derive $(\mathbf{DS}.k{+}1, n, N)$ from $(\mathbf{DS}.k, n', N)$ for any $1 \leq n' \leq n$, i.e., to prove the following statement:

Theorem 6 (scale induction step)**.** *Let* $\mathbf{H}^{(N)}(\omega) = -\mathbf{\Delta} + \sum_{j=1}^{N} V(x_j,\omega) + \mathbf{U}$*, where* $\mathbf{\Delta}$*,* $\mathbf{U}$*,* V *satisfy* $(\mathbf{K})$*,* $(\mathbf{I})$*,* $(\mathbf{P1})$*, respectively.*

Then there exists $0 < L^* < \infty$ *such that, if* $L_0 \geq L^*$*, then, if for* $k \geq 0$*, property* $(\mathbf{DS}.k, n', N)$ *holds true for any* $1 \leq n' \leq n \leq N$*, then* $(\mathbf{DS}.k+1, n, N)$ *holds true.*

Proof. This theorem splits into Theorem 7, Theorem 8, and Theorem 9. The proofs of these theorems occupy Section 4. Their end will mark also the end of the proof of Theorem 1 (A) under $(\mathbf{P1})$. ☐

4. Multiscale induction under (K), (I) and (P1)

In section 4.1, we establish some useful geometrical facts valid for any $n \geq 1$. Starting from section 4.2, we perform the induction step $n-1 \rightsquigarrow n$, using scale induction in L_k, assuming all necessary properties of n'-particle systems with any $1 \leq n' \leq n-1$ and in cubes of any size L_j, $j \geq 0$. Further, the induction step $L_k \rightsquigarrow L_{k+1}$ is carried out separately for three types of pairs of singular n-particle cubes (see sections 4.2-4.4). At this step, we are allowed also to assume property $(\mathbf{DS}.k, n, N)$ of N-particle systems in cubes of size L_k, and aim to prove $(\mathbf{DS}.k+1, n, N)$ in cubes of size L_{k+1}.

As indicated by the title of this section, we assume here the condition $(\mathbf{P1})$ which gives rise to a simpler MPMSA scheme than in [6]. The key point is a more convenient eigenvalue concentration bound (3.5) (see Lemma 1) established earlier by Chulaevsky [4]. However, this bound has not been proven so far for a more general class of i.i.d. potentials with log-Hölder continuous marginal distribution. For this reason, we provide in Section 5 an adaptation of the original MPMSA scheme used in [6] (for strongly disordered systems) to the localization at low energies, under a weaker assumption $(\mathbf{P2})$.

4.1. **Fully and partially interactive cubes.** Consider the diagonal in $\mathbb{Z}^{nd}$, $n \geq 1$:

$$\mathbb{D} = \left\{\mathbf{x} = (x, \ldots, x) \in \mathbb{Z}^{nd} : x \in \mathbb{Z}^d\right\}. \tag{4.1}$$

Definition 6 (fully/partially interactive)**.** An n-particle cube $\mathbf{C}^{(n)}_L(\mathbf{u}) \subset \mathbb{Z}^{nd}$ is

(i) fully interactive (FI) if $\mathrm{dist}(\mathbf{u}, \mathbb{D}) \leq 2n(L + r_0)$,
(ii) partially interactive (PI) otherwise, i.e., if $\mathrm{dist}(\mathbf{u}, \mathbb{D}) > 2n(L + r_0)$.

The following simple statement clarifies the notion of PI cube.

Lemma 8. *If* $\operatorname{dist}(\mathbf{u}, \mathbb{D}) > 2n(L + r_0)$, *then there exists a subset* $\mathcal{J} \subset \{1, \ldots, n\}$ *with* $1 \leq \operatorname{card} \mathcal{J} \leq n - 1$ *such that*

$$\max_{\substack{j_1 \in \mathcal{J} \\ j_2 \notin \mathcal{J}}} |u_{j_1} - u_{j_2}| > r_0. \tag{4.2}$$

Consequently, if a cube $\mathbf{C}_L^{(n)}(\mathbf{u})$ *is PI, then there exists a subset* $\mathcal{J} \subset \{1, \ldots, n\}$ *with* $1 \leq \operatorname{card} \mathcal{J} \leq n - 1$ *such that*

$$\Pi_{\mathcal{J}} \mathbf{C}_L^{(n)}(\mathbf{u}) \cap \Pi_{\mathcal{J}^c} \mathbf{C}_L^{(n)}(\mathbf{u}) = \varnothing.$$

In other words, setting $n' = |\mathcal{J}| \geq 1$, $n'' = |\mathcal{J}^c| \geq 1$, $\mathbf{u}' = \mathbf{u}_{\mathcal{J}} \in \mathbb{Z}^{dn'}$, and $\mathbf{u}'' = \mathbf{u}_{\mathcal{J}^c} \in \mathbb{Z}^{dn''}$, one can represent $\mathbf{C}_L^{(n)}(\mathbf{u})$ as a cartesian product

$$\mathbf{C}_L^{(n)}(\mathbf{u}) = \mathbf{C}_L^{(n')}(\mathbf{u}') \times \mathbf{C}_L^{(n'')}(\mathbf{u}'') \tag{4.3}$$

with disjoint projections

$$\Pi_{\mathcal{J}} \mathbf{C}_L^{(n)}(\mathbf{u}) = \Pi \mathbf{C}_L^{(n')}(\mathbf{u}') \subset \mathbb{Z}^d,$$
$$\Pi_{\mathcal{J}^c} \mathbf{C}_L^{(n)}(\mathbf{u}) = \Pi \mathbf{C}_L^{(n'')}(\mathbf{u}'') \subset \mathbb{Z}^d.$$

For a given PI cube, there can exist several decompositions of the form (4.3). We will assume that such a decomposition is associated with each PI cube and call it the "canonical" decomposition.

Now we turn to geometrical properties of FI cubes.

Lemma 9. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^d \times \cdots \times \mathbb{Z}^d \simeq \mathbb{Z}^{nd}$. *Then*

$$\operatorname{dist}(\mathbf{x}, \mathbb{D}) \leq \max_{i,j} |x_i - x_j| = \operatorname{diam} \Pi \mathbf{x} \leq 2 \operatorname{dist}(\mathbf{x}, \mathbb{D}). \tag{4.4}$$

Proof. By definition of $\mathbb{D}$,

$$\begin{aligned}
\operatorname{dist}(\mathbf{x}, \mathbb{D}) &= \min_{u \in \mathbb{Z}^d} \max_j |x_j - u| \\
&\leq \min_i \max_j |x_j - x_i| \qquad\qquad (\text{each } x_i \text{ is a particular } u \in \mathbb{Z}^d) \\
&\leq \max_{i,j} |x_j - x_i| \\
&= \operatorname{diam}\{x_1, \ldots, x_n\} = \operatorname{diam} \Pi \mathbf{x},
\end{aligned}$$

proving the first inequality. On the other hand, if $\mathbf{u} = (u, \ldots, u) \in \mathbb{D} \subset \mathbb{Z}^{nd}$ is the closest point to $\mathbf{x}$ in $\mathbb{D}$, then, for any $1 \leq i, j \leq n$,

$$|x_i - x_j| \leq |x_i - u| + |u - x_j| \leq 2 \operatorname{dist}(\mathbf{x}, \mathbf{u}) = 2 \operatorname{dist}(\mathbf{x}, \mathbb{D}). \qquad \square$$

Lemma 10. *Let* $n \geq 1$, $L > 8r_0$ *and consider two n-particle fully interactive cubes* $\mathbf{C}_L^{(n)}(\mathbf{u})$ *and* $\mathbf{C}_L^{(n)}(\mathbf{v})$ *with* $|\mathbf{x} - \mathbf{y}| > 11\, nL$. *Then*

$$\Pi \mathbf{C}_L^{(n)}(\mathbf{u}) \cap \Pi \mathbf{C}_L^{(n)}(\mathbf{v}) = \varnothing. \tag{4.5}$$

Proof. If for some $R > 0$,

$$R < |\mathbf{x} - \mathbf{y}| = \max_{1 \leq j \leq n} |x_j - y_j|,$$

then there exists $1 \leq j_0 \leq n$ such that $|x_{j_0} - y_{j_0}| > R$. Since both cubes are fully interactive, $\mathrm{dist}(\mathbf{x}, \mathbb{D}) \leq 2n(L + r_0)$, $\mathrm{dist}(\mathbf{y}, \mathbb{D}) \leq 2n(L + r_0)$, so we can use (4.4) for the centers $\mathbf{x}$, $\mathbf{y}$ and write:

$$\begin{aligned} |x_{j_0} - x_i| &\leq 2 \times 2n(L + r_0), \\ |y_{j_0} - y_j| &\leq 2 \times 2n(L + r_0). \end{aligned}$$

By triangle inequality, for any $1 \leq i, j \leq n$ and $R > 11\,nL > 10\,nL + 8\,nr_0$, we have

$$\begin{aligned} |x_i - y_j| &\geq |x_{j_0} - y_{j_0}| - |x_{j_0} - x_i| - |y_{j_0} - y_j| \\ &> 10nL + 8nr_0 - 8n(L + r_0) = 2nL. \end{aligned}$$

Therefore, for any $1 \leq i, j \leq n$,

$$\min_{i,j} \mathrm{dist}(C_L(x_i), C_L(y_j)) \geq \min_{i,j} |x_i - y_j| - 2L > 2(n-1)L \geq 0.$$

This means that

$$\mathit{\Pi}\mathbf{C}_L^{(n)}(\mathbf{x}) \cap \mathit{\Pi}\mathbf{C}_L^{(n)}(\mathbf{y}) = \bigcup_{i,j} (C_L(x_i) \cap C_L(y_j)) = \varnothing. \qquad \square$$

It is clear from the proof that the condition $|\mathbf{x} - \mathbf{y}| > 11\,nL$ could have been replaced by a more explicit, and sharper, inequality $|\mathbf{x} - \mathbf{y}| > 10\,nL + 8\,nr_0$, but we chose the first inequality for simplicity.

Given $1 \leq n \leq N$, and assuming $(\mathbf{DS}.k, n', N)$ for any $1 \leq n' \leq n$, we will prove $(\mathbf{DS}.k+1, n, N)$, separately for the following three types of pairs of cubes:

(I) $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{x})$ and $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{y})$ are both PI-cubes.

(II) $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{x})$ and $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{y})$ are both FI-cubes.

(III) One of the cubes $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{x})$ or $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{y})$ is PI, while the other is FI.

In the rest of the paper we will denote by I the interval $[E^0, E^*]$.

4.2. **Pairs of partially interactive cubes.** We consider here a pair of L_{k+1}-distant partially interactive cubes $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{x})$ and $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{y})$ and prove that $(\mathbf{DS}.k{+}1, n, N)$ hold for such a pair. We are allowed to assume property $(\mathbf{DS}.k, n', N)$ for any $1 \leq n' < n$ and every pair of L_k-distant cubes $\mathbf{C}_{L_k}^{(n')}(\mathbf{u})$ and $\mathbf{C}_{L_k}^{(n')}(\mathbf{v})$.

Let $\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{u})$ be a PI-cube with the canonical decomposition (4.3). We also write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ for any point $\mathbf{x} \in \mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{u})$, in the same way as $(\mathbf{u}', \mathbf{u}'')$. In fact owing to the symmetry of the operator $\mathbf{U}$, we can simply take without loss of generality $\mathcal{J} = \{1, \dots, n'\}$ and $\mathcal{J}^{\mathrm{c}} = \{n'+1, \dots, n\}$ so that the corresponding Hamiltonian $\mathbf{H}_{\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{u})}$ is written in the form:

$$\mathbf{H}^{(n)}_{\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{u})} \mathbf{\Psi}(\mathbf{x}) = (-\mathbf{\Delta}\mathbf{\Psi})(\mathbf{x}) + [\mathbf{U}(\mathbf{x}') + \mathbf{V}(\mathbf{x}', \omega) + \mathbf{U}(\mathbf{x}'') + \mathbf{V}(\mathbf{x}'', \omega)]\, \mathbf{\Psi}(\mathbf{x}) \quad (4.6)$$

or, in compact form

$$\mathbf{H}^{(n)}_{\mathbf{C}_{L_{k+1}}^{(n)}(\mathbf{u})} = \mathbf{H}^{(n')}_{\mathbf{C}_{L_{k+1}}^{(n')}(\mathbf{u}')} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{H}^{(n'')}_{\mathbf{C}_{L_{k+1}}^{(n'')}(\mathbf{u}'')}.$$

Here $\mathbf{H}^{(n')}_{\mathbf{C}_{L_{k+1}}^{(n')}(\mathbf{u}')}$ and $\mathbf{H}^{(n'')}_{\mathbf{C}_{L_{k+1}}^{(n'')}(\mathbf{u}'')}$ are the Hamiltonians acting on $\mathbf{x}'$ and $\mathbf{x}''$, respectively, while $\mathbf{I}$ is the identity operator on the complementary variable. We denote by

$\mathbf{G}^{(n')}(\mathbf{u}',\mathbf{v}';E)$ and $\mathbf{G}^{(n'')}(\mathbf{u}'',\mathbf{v}'';E)$ the corresponding Green functions, respectively. Let

$$\{(\lambda_i,\varphi_i)\}_{i=1,\dots,|\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{u}')|} \quad \text{and} \quad \{(\mu_j,\phi_j)\}_{j=1,\dots,|\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{u}'')|}$$

be the eigenvalues and corresponding eigenfunctions of $\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{u}')}$ and $\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{u}'')}$, respectively. Then we can choose eigenvectors $\mathbf{\Psi}_{ij}$ of $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}$ as tensor products

$$\mathbf{\Psi}_{ij}(\mathbf{x}) = \varphi_i(\mathbf{x}') \otimes \phi_j(\mathbf{x}'') \tag{4.7}$$

while corresponding eigenvalues

$$E_{ij} = \lambda_i + \mu_j. \tag{4.8}$$

The eigenvectors of finite volume Hamiltonians appearing in subsequent arguments and calculations will be assumed normalized. Introduce the following

Definition 7 (m-tunnelling)**.** Let $1 \le n \le N$, $k \ge 1$, $I = [E^0, E^*]$ and $m > 0$ be fixed.

(i) A cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is called *m-tunnelling* (m-T) if there exist $E \in I$ and two L_{k-1}-distant cubes $\mathbf{C}^{(n)}_{L_{k-1}}(\mathbf{v}_1)$, $\mathbf{C}^{(n)}_{L_{k-1}}(\mathbf{v}_2) \subset \mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ which are both (E,m)-S.

(ii) A cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is called *(m,n',n'')-partially tunnelling* ((m,n',n'')-PT) if it has a factorization $\mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \times \mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')$ in which either $\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')$ or $\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')$ is m-tunnelling. Otherwise it is called (m,n',n'')-NPT.

(iii) A cube $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is called *m-partially tunnelling* (m-PT) if it is (m,n',n'')-PT for some $n',n'' \ge 1$ with $n'+n''=N$. Otherwise, it is called m-NPT.

The following statement is borrowed from [6].

Lemma 11. *Let $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ be a PI cube of the form*

$$\mathbf{C}^{(n)}_{L_k}(\mathbf{u}) = \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \times \mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')$$

where $\mathbf{u} = (\mathbf{u}',\mathbf{u}'')$ with $\mathbf{u}' = (u_1,\dots,u_{n'}) \in \mathbb{Z}^{n'd}$ and $\mathbf{u}'' = (u_{n'+1},\dots,u_n) \in \mathbb{Z}^{n''d}$. We assume that

(i) *$|u_{j_1} - u_{j_2}| > 2L_k + r_0$ for all $1 \le j_1 \le n'$, $n'+1 \le j_2 \le n$.*
(ii) *$\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is m-NPT for some $m > 0$.*
(iii) *$\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ is E-CNR for some $E \in \mathbb{R}$.*

Then,

$$\max_{\substack{1 \le i \le |\mathbf{C}^{(n')}_{L_k}| \\ \mathbf{v}'' \in \partial^- \mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}} \left|\mathbf{G}^{(n'')}(\mathbf{u}'',\mathbf{v}'';E-\lambda_i)\right| \le \mathrm{e}^{-m'L_k} \tag{4.9}$$

$$\max_{\substack{1 \le j \le |\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')| \\ \mathbf{v}' \in \partial^- \mathbf{C}^{(n')}_{L_k}(\mathbf{u}')}} \left|\mathbf{G}^{(n')}(\mathbf{u}',\mathbf{v}';E-\mu_j)\right| \le \mathrm{e}^{-m'L_k}. \tag{4.10}$$

where

$$m' = \gamma(m, L_k, n-1)\left(1 - L_k^{-(1-\beta)} - L_k^{-1} \ln L_k^{(n-1)d}\right) \tag{4.11}$$

and where $\{\lambda_i\}$ and $\{\mu_j\}$ are the eigenvalues of $\mathbf{H}^{(n')}_{\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')}$ and $\mathbf{H}^{(n'')}_{\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}$, respectively.

Proof. See [6, Lemma 3]. $\square$

Lemma 12. *A PI cube* $\mathbf{C}^{(n)}_{L_k}(\mathbf{u})$ *which satisfies assumptions* (i)-(ii)-(iii) *of Lemma 11 is* (E,m)*-NS.*

Proof. Using Definition (2.16) which gives the explicit form of $\gamma(m,L_k,N)$, it is easy to see that $m' > \gamma(m,L_k,n)$, hence inequalities (4.9) and (4.10) of Lemma 11 imply

$$\max_{\substack{1\le i\le |\mathbf{C}^{(n')}_{L_k}| \\ \mathbf{v}''\in\partial^-\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')}} \left|\mathbf{G}^{(n'')}(\mathbf{u}'',\mathbf{v}'';E-\lambda_i)\right| \le \mathrm{e}^{-\gamma(m,L_k,n)L_k}, \tag{4.12}$$

$$\max_{\substack{1\le j\le |\mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')| \\ \mathbf{v}'\in\partial^-\mathbf{C}^{(n')}_{L_k}(\mathbf{u}')}} \left|\mathbf{G}^{(n')}(\mathbf{u}',\mathbf{v}';E-\mu_j)\right| \le \mathrm{e}^{-\gamma(m,L_k,n)L_k}. \tag{4.13}$$

Then $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}) = \mathbf{C}^{(n')}_{L_k}(\mathbf{u}') \times \mathbf{C}^{(n'')}_{L_k}(\mathbf{u}'')$ is (E,m)-NS, using the same representations

$$\mathbf{G}^{(n)}(\mathbf{u},\mathbf{v};E) = \sum_i \varphi_i(\mathbf{u}')\varphi_i(\mathbf{v}')\mathbf{G}^{(n'')}(\mathbf{u}'',\mathbf{v}'';E-\lambda_i),$$
$$\mathbf{G}^{(n)}(\mathbf{u},\mathbf{v};E) = \sum_j \psi_j(\mathbf{u}'')\psi_j(\mathbf{v}'')\mathbf{G}^{(n')}(\mathbf{u}',\mathbf{v}';E-\mu_j)$$

as in [6, Appendix B]. $\square$

Lemma 13. *Let* $2\le n\le N$. *We assume property* $(\mathbf{DS}.k,n',N)$ *for any* $1\le n'<n$. *Let* $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ *be a PI cube and let*

$$\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) = \mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}') \times \mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{y}'')$$

be a factorization, $\mathbf{y}=(\mathbf{y}',\mathbf{y}'')$, $\mathbf{y}'=(y_1,\dots,y_{n'})\in\mathbb{Z}^{n'd}$, $\mathbf{y}''=(y_{n'+1},\dots,y_N)\in\mathbb{Z}^{n''d}$ *with* $n=n'+n''$, $n',n''\ge 1$ *and*

$$\min_{\substack{1\le i\le n' \\ n'+1\le j\le N}} |y_i-y_j| > 2L_k + r_0. \tag{4.14}$$

Then

$$\mathbb{P}\{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } m\text{-}PT\} \le \frac{1}{4}L_{k+1}^{-2p\,4^{N-n}}. \tag{4.15}$$

Proof. By definition 7, we have

$$\mathrm{T}_{\mathbf{y}} := \left\{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } m\text{-PT}\right\} \subset \left\{\text{either } \mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}') \text{ or } \mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{y}'') \text{ is } m\text{-T}\right\}.$$

Since $n'<n$ we can use property $(\mathbf{DS}.k,n',N)$. Thus by combining this property with an upper bound of the number of possible pairs of centres $\mathbf{v}_1$, $\mathbf{v}_2\in\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}')$, we get:

$$\mathbb{P}\left\{\mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}') \text{ is } m\text{-T}\right\} \le \frac{3^{2n'd}}{2}L_{k+1}^{2n'd}\times L_k^{-2p4^{N-n'}}. \tag{4.16}$$

Similarly for $\mathbb{P}\{\mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{y}'')$ is m-T$\}$, using $(\mathbf{DS}.k, n'', N)$ which is known since $n'' < n$. Therefore,

$$\begin{aligned}
\mathbb{P}\{\mathrm{T}_{\mathbf{y}}\} &\le \mathbb{P}\{\text{either } \mathbf{C}^{(n')}_{L_{k+1}}(\mathbf{y}') \text{ or } \mathbf{C}^{(n'')}_{L_{k+1}}(\mathbf{y}'') \text{ is } m\text{-PT}\}\\
&\le C(n,N,d) L_{k+1}^{-2p\,\frac{4^{N-(n-1)}}{\alpha}+2(n-1)d}\\
&\le C(n,N,d) L_{k+1}^{-2p\,\frac{4}{\alpha}\,4^{N-n}+2(n-1)d}\\
&\le C(n,N,d) L_{k+1}^{-\frac{8}{3}2p\,4^{N-n}+2(n-1)d} && (\alpha = 3/2)\\
&\le \frac{1}{2} L_{k+1}^{-4p\,4^{N-n}} && (4.17)\\
&\le \frac{1}{4} L_{k+1}^{-2p\,4^{N-n}}
\end{aligned}$$

for large enough L_0, since $p > 4\alpha N d = 6Nd$ and $q > 4^N p$ by our assumption. □

Theorem 7. *There exists $L_1^* > 0$ such that if $L_0 \ge L_1^*$ and if for $k \ge 0$ and $1 \le n \le N$, $(\mathbf{DS}.k, n', N)$ holds true for any $1 \le n' < n$, then $(\mathbf{DS}.k+1, n, N)$ holds true*

(i) *for any pair of L_{k+1}-distant PI cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$.*

Proof. Let $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ be two L_{k+1}-distant PI-cubes. We consider the events:

$$\begin{aligned}
\mathrm{B}_{k+1} &= \{\exists\, E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ and } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ are } (E,m)\text{-S}\},\\
\mathrm{R} &= \{\exists\, E \in I : \text{neither } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } E\text{-CNR}\},\\
\mathrm{T}_{\mathbf{x}} &= \{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ is } m\text{-PT}\},\\
\mathrm{T}_{\mathbf{y}} &= \{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } m\text{-PT}\}.
\end{aligned}$$

If $\omega \in \mathrm{B}_{k+1} \setminus \mathrm{R}$, then $\forall E \in I$, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ or $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR. If $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR, then it must be m-PT: otherwise it would have been (E,m)-NS by Lemma 12. Similarly, if $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ is E-CNR, then it must be m-PT. This implies that

$$\mathrm{B}_{k+1} \subset \mathrm{R} \cup \mathrm{T}_{\mathbf{x}} \cup \mathrm{T}_{\mathbf{y}}.$$

Therefore,

$$\begin{aligned}
\mathbb{P}\{\mathrm{B}_{k+1}\} &\le \mathbb{P}\{\mathrm{R}\} + \mathbb{P}\{\mathrm{T}_{\mathbf{x}}\} + \mathbb{P}\{\mathrm{T}_{\mathbf{y}}\}\\
&\le L_{k+1}^{-q} + 2\mathbb{P}\{\mathrm{T}_{\mathbf{x}}\}\\
&\le L_{k+1}^{-4p\,4^{N-n}} + L_{k+1}^{-4p\,4^{N-n}} && (q > 4^N p)\\
&\le L_{k+1}^{-2p\,4^{N-n}}
\end{aligned}$$

where we used $(\mathbf{WS2}.n)$ to estimate $\mathbb{P}\{\mathrm{R}\}$ and (4.17) to estimate $\mathbb{P}\{\mathrm{T}_{\mathbf{x}}\}$. □

Given an n-particle cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ and $E \in \mathbb{R}$, we denote by $M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E)$ the maximal number of pairwise L_k-distant, *partially interactive* (E,m)-singular cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$. Further, set

$$M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), I) := \sup_{E \in I} M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E). \tag{4.18}$$

Similarly, we denote by $M_{\rm D}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E)$ the maximal number of pairwise L_k-distant, *fully interactive* (E,m)-singular cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)})\subset\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ and set

$$M_{\rm D}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),I):=\sup_{E\in I}M_{\rm D}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),E). \tag{4.19}$$

Lemma 14. *With the above notations,*

$$\mathbb{P}\left\{M_{\rm ND}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),I)\geq 2\right\}\leq\frac{3^{nd}}{2}L^{2nd}_{k+1}\left(L_k^{-q}+L_k^{-4p\,4^{N-n}}\right). \tag{4.20}$$

Proof. The number of possible pairs of centres $\{\mathbf{u}^{(j_1)},\mathbf{u}^{(j_2)}\}$ of L_k-distant cubes

$$\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j_1)}),\ \mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j_1)})\subset\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$$

which are PI and (E,m)-singular is bounded by $\frac{3^{nd}}{2}L^{2nd}_{k+1}$, then

$$\mathbb{P}\left\{M_{\rm ND}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),I)\geq 2\right\}\leq\frac{3^{nd}}{2}L^{2nd}_{k+1}\times\mathbb{P}\left\{\mathrm{B}_k\right\}$$

with $\mathbb{P}\left\{\mathrm{B}_k\right\}\leq L_k^{-q}+2\mathbb{P}\left\{\mathrm{T}_{\mathbf{x}}\right\}$ and $\mathbb{P}\left\{\mathrm{T}_{\mathbf{x}}\right\}\leq\frac{1}{2}L_k^{-4p\,4^{N-n}}$ by (4.17). $\square$

4.3. **Pairs of fully interactive cubes.** Our aim now is to prove $(\mathbf{DS}.k+1,n,N)$ for a pair of L_{k+1}-distant fully interactive cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$. We always assume property $(\mathbf{DS}.k,n,N)$ for any pair of L_k-distant cubes.

The main result of this section is Theorem 8. We need some preliminary results.

Lemma 15. *Given $k\geq 0$, assume that property $(\mathbf{DS}.k,n,N)$ holds true for all pairs of L_k-distant fully interactive cubes. Then for any $\ell\geq 1$*

$$\mathbb{P}\left\{M_{\rm D}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}),I)\geq 2\ell\right\}\leq C(n,N,d,\ell)L_k^{2\ell dn\alpha}L_k^{-2\ell p\,4^{N-n}}. \tag{4.21}$$

Proof. Suppose there exist 2ℓ pairwise L_k-distant, fully interactive cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)})\subset\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$, $1\leq j\leq 2\ell$. Then, by Lemma 10, for any pair $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i-1)})$, $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i)})$, the corresponding random Hamiltonians $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i-1)})}$ and $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i)})^{(n)}}$ are independent, and so are their spectra and their Green functions. Moreover, the pairs of operators

$$\left(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i-1)})}(\omega),\,\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i)})}(\omega)\right),$$

$i=1,\dots,\ell$, form an independent family. Each operator $\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(i)})}(\omega)$, $i=1,\dots,2\ell$ is indeed measurable with respect to the σ-algebra $\mathcal{B}_i$ generated by the random variables $\{V(x,\omega):x\in\Pi\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(i)})\}$. By Lemma 10, the projections $\Pi\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(i)})$, $i=1,\dots,2\ell$ are pairwise disjoint, so that all sigma-algebras $\mathcal{B}_i$, $i=1,\dots,2\ell$ are independent. Thus any collection of events relative to the pairs $\left(\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i-1)})}(\omega),\mathbf{H}^{(n)}_{\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i)})}(\omega)\right)$, $i=1,\dots,\ell$, also form an independent family. For $i=1,\dots,\ell$ we consider the events:

$$\mathrm{A}_i=\left\{\exists\,E\in I:\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i-1)})\text{ and }\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(2i+2)})\text{ are }(E,m)\text{-S}\right\}.$$

Then by assumption $(\mathbf{DS}.k,n,N)$, we have, for $i=1,\dots,\ell$,

$$\mathbb{P}\left\{\mathrm{A}_i\right\}\leq L_k^{-2p\,4^{N-n}}, \tag{4.22}$$

and, by independence of events $\mathrm{A}_1, \ldots, \mathrm{A}_\ell$,

$$\mathbb{P}\Big\{ \bigcap_{1\le i\le \ell} \mathrm{A}_i \Big\} = \prod_{i=1}^{\ell} \mathbb{P}(\mathrm{A}_i) \le \big(L_k^{-2p}\big)^{\ell\, 4^{N-n}}. \tag{4.23}$$

To complete the proof, note that the total number of different families of 2ℓ cubes $\mathbf{C}^{(n)}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$, $1 \le j \le 2\ell$, is bounded by

$$\frac{1}{(2\ell)!}\left|\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})\right|^{2\ell} \le C(n, N, \ell, d) L_k^{2\ell d n\alpha}. \qquad \square$$

Lemma 16 ([6, Lemma 12]). *Fix an odd positive integer J and suppose further that*

(i) $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ *is E-CNR,*

(ii) $M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E) + M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u}), E) \le J$.

Then for L_0 large enough,

$$\max_{\mathbf{v}\in\partial^- \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})} \left| \mathbf{G}^{(n)}_{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E) \right| \le \mathrm{e}^{-m' L}$$

where, for some $C \in (0, +\infty)$,

$$m' \ge \gamma(m, L_k, n)\big(1 - C L_k^{-1/2}\big). \tag{4.24}$$

Using again the definition (2.16) of the quantity $\gamma(m, L, n)$, one can easily derive from Lemma 16 the following bound.

Lemma 17. *Under assumptions and with notations of Lemma 16, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{u})$ is (E, m)-NS.*

Proof. Corollary of Lemma 16. Note that (4.24) implies

$$m' \ge \gamma(m, L_{k+1}, n) \tag{4.25}$$

provided that L_0 is large enough. $\square$

Theorem 8. *There exists $L_2^* > 0$ such that if $L_0 \ge L_2^*$ and if for $k \ge 0$ and $1 \le n \le N$, $(\mathbf{DS}.k, n, N)$ holds true, then $(\mathbf{DS}.k+1, n, N)$ holds true*

(ii) *for any pair of L_{k+1}-distant FI cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$.*

Proof. Consider a pair of L_{k+1}-distant cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ and set

$$\begin{aligned}
\mathrm{B}_{k+1} &= \Big\{\exists\, E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ and } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ are } (E, m)\text{-S}\Big\},\\
\Sigma &= \Big\{\exists\, E \in I : \text{neither } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } E\text{-CNR}\Big\},\\
\mathrm{S}_{\mathbf{x}} &= \Big\{\exists\, E \in I : M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) + M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \ge J + 1\Big\},\\
\mathrm{S}_{\mathbf{y}} &= \Big\{\exists\, E \in I : M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}), E) + M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}), E) \ge J + 1\Big\}.
\end{aligned}$$

Let $\omega \in \mathrm{B}_{k+1}$, assume that $\omega \notin \Sigma \cup \mathrm{S}_{\mathbf{x}}$, then $\forall E \in I$ either $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ or $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR and $M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) + M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), E) \le J$. Thus the cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$

cannot be E-CNR: indeed, by Lemma 17 it would be (E,m)-NS. So the cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ which is E-CNR and (E,m)-S. This implies again by Lemma 17 that

$$M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}),E)+M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}),E)\geq J+1.$$

Therefore $\omega\in\mathrm{S}_{\mathbf{y}}$, so that

$$\mathrm{B}_{k+1}\subset\Sigma\cup\mathrm{S}_{\mathbf{x}}\cup\mathrm{S}_{\mathbf{y}}.$$

Hence,

$$\begin{aligned}\mathbb{P}\left\{\mathrm{B}_{k+1}\right\}&\leq\mathbb{P}\{\Sigma\}+\mathbb{P}\{\mathrm{S}_{\mathbf{x}}\}+\mathbb{P}\{\mathrm{S}_{\mathbf{y}}\}\\&\leq L_{k+1}^{-q}+2\,\mathbb{P}\{\mathrm{S}_{\mathbf{x}}\}\\&\leq L_{k+1}^{-4p}+2\,\mathbb{P}\{\mathrm{S}_{\mathbf{x}}\}.\end{aligned}$$

Now let us estimate $\mathbb{P}\{\mathrm{S}_{\mathbf{x}}\}$. Set $J=2\ell+1$, with $\ell=2$ then the inequality

$$M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}),E)+M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}),E)\geq 2\ell+2$$

implies that either $M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}),E)\geq 2$, or $M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}),E)\geq 2\ell$. Therefore, by Lemma 14 and Lemma 15,

$$\begin{aligned}\mathbb{P}\{\mathrm{S}_{\mathbf{x}}\}&\leq\mathbb{P}\left\{\exists\,E\in I:M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}),E)\geq 2\right\}\\&\quad+\mathbb{P}\left\{\exists\,E\in I:M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}),E)\geq 2\ell\right\}\\&\leq\frac{3^{2Nd}}{2}L_{k+1}^{2nd}(L_k^{-q}+L_k^{-4p\,4^{N-n}})+C(n,N,\ell,d)L_{k+1}^{2\ell dn-\frac{2\ell p}{\alpha}4^{N-n}}\\&\leq C(n,N,d)\left(L_{k+1}^{-\frac{4^Np}{\alpha}+2nd}+L_{k+1}^{-\frac{4p}{\alpha}4^{N-n}+4nd}\right)\\&\leq C(n,N,d)\left(L_{k+1}^{-\frac{4p}{\alpha}4^{N-1}+2nd}+L_{k+1}^{-\frac{4p}{\alpha}4^{N-n}+4nd}\right)\\&\leq C(n,N,d)\left(L_{k+1}^{-\frac{4p}{\alpha}4^{N-n}+2nd}+L_{k}^{-\frac{4p}{\alpha}4^{N-n}+4nd}\right)\qquad(\ell=2,\ \alpha=3/2)\\&\leq\frac{1}{4}L_{k+1}^{-2p\,4^{N-n}}\end{aligned}$$

where we used $q>4^Np$ and $p>4\alpha Nd=6Nd$. ☐

4.4. **Mixed pairs of cubes.** Now it remains only to derive $(\mathbf{DS}.k+1,n,N)$ in case (III), i.e., for pairs of n-particle cubes where one of the cube is partially interactive while the other is fully interactive.

Theorem 9. *There exists $L_3^*>0$ such that if $L_0\geq L_3^*$ and if for $k\geq 0$ and $1\leq n\leq N$, $(\mathbf{DS}.k,n',N)$ holds true for any $1\leq n'\leq n$, then $(\mathbf{DS}.k+1,n,N)$ holds true*

(iii) *for any pair of L_{k+1}-distant cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ where one is partially interactive while the other is fully interactive.*

Proof. Consider a pair of L_{k+1}-distant N-particle cubes $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$, $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ and suppose that $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ is partially interactive while $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is fully interactive. Introduce

the events

$$\begin{aligned}
\mathrm{B}_{k+1} &= \left\{\exists\, E \in I : \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ and } \mathbf{C}^{(n)}_{L_{k+1}} \text{ are } (E,m)\text{-S}\right\},\\
\Sigma &= \left\{\exists\, E \in I : \text{neither } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}) \text{ is } E\text{-CNR}\right\},\\
\mathrm{T}_{\mathbf{x}} &= \left\{\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}) \text{ is } m\text{-T}\right\},\\
\mathrm{S}_{\mathbf{y}} &= \left\{\exists\, E \in I : M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}), E) + M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y}), E) \ge J+1\right\}.
\end{aligned}$$

Let $\omega \in \mathrm{B}_{k+1} \setminus (\Sigma \cup \mathrm{T}_{\mathbf{x}})$, then for all $E \in I$ either $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ or $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR and $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ is m-NT. The cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x})$ cannot be E-CNR. Indeed, by Lemma 12 it would have been (E,m)-NS. Thus the cube $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ is E-CNR, so by Lemma 17, $M_{\mathrm{ND}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), I) + M_{\mathrm{D}}(\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{x}), I) \ge J+1$: otherwise $\mathbf{C}^{(n)}_{L_{k+1}}(\mathbf{y})$ would be (E,m)-NS. Therefore $\omega \in \mathrm{S}_{\mathbf{y}}$. Consequently,

$$\mathrm{B}_{k+1} \subset \Sigma \cup \mathrm{T}_{\mathbf{x}} \cup \mathrm{S}_{\mathbf{y}}.$$

Recall that the probabilities $\mathbb{P}\{\mathrm{T}_{\mathbf{x}}\}$ and $\mathbb{P}\{\mathrm{S}_{\mathbf{y}}\}$ have already been estimated in Sections 4.2 and 4.3 under the condition that $(\mathbf{DS}.k, n', N)$ is known for any $1 \le n' \le n$. Now, we obtain

$$\begin{aligned}
\mathbb{P}\left\{\mathrm{B}_{k+1}\right\} &\le \mathbb{P}\{\Sigma\} + \mathbb{P}\{\mathrm{T}_{\mathbf{x}}\} + \mathbb{P}\{\mathrm{S}_{\mathbf{y}}\}\\
&\le L_{k+1}^{-q} + \frac{1}{4} L_{k+1}^{-2p\,4^{N-n}} + \frac{1}{4} L_{k+1}^{-2p\,4^{N-n}}\\
&\le L_{k+1}^{-2p\,4^{N-n}}. \qquad \square
\end{aligned}$$

4.5. **Proof of Theorem 1 (A) under (P1).** Assuming property $(\mathbf{DS}.k, n', N)$ for any $1 \le n' \le n$ we have proven $(\mathbf{DS}.k+1, n, N)$

(i) for any pair of L_{k+1}-distant PI cubes (Theorem 7),
(ii) for any pair of L_{k+1}-distant FI cubes (Theorem 8),
(iii) for any mixed pair of L_{k+1}-distant cubes (Theorem 9),

i.e., we have proven $(\mathbf{DS}.k+1, n, N)$ for any pair of L_{k+1}-distant cubes: Theorem 6 is proven. Since the initial scale estimate (for L_0) has been established in Theorem 5, both theorems imply Theorem 3 by induction in k, hence assertion (A) of Theorem 1.

5. Proof of localization under (K), (I) and (P2)

Now we will establish multi-particle Anderson localization at low energies under weaker assumptions on the external random potentials. Namely, we will not use the assumption (**P1**), the validity of which has been established so far for a more restricted class of random fields than (**P2**).

To this end, we will provide an adaptation of the original scheme proposed in [6]. However, this results in more complicated geometrical arguments and less optimal bounds on the decay of eigenfunctions. In particular, an eigenvalue concentration bound similar to Lemma 1 will not be established for *all* sufficiently distant pairs of cubes, but only for those satisfying an additional condition ("separable pairs"). This difficulty was encountered in [6] and in [1]. In the latter work, Aizenman and Warzel explicitly analyzed the nature of this technical problem and explained why they had to replace norm-distance estimates of the eigenfunction decay by those using the so-called Hausdorff distance; see the discussion in [1, §1.3]. A solution to this problem

has been proposed in [4], where the bound of the form (**WS2**.n) was proven under the assumption (**P1**), resulting in a streamlined description of the decay of Green functions and of eigenfunction.

On the other hand, an adaptation of the multi-particle MSA scheme proposed in [6] to the proof of localization at low energies is fairly straightforward. Indeed, in the framework of the multi-particle MSA, like in the case of single-particle models, the role of the assumptions such as

- strong disorder: $|g| \gg 1$, or
- low energies: $E \in [E^0, E^0 + \eta]$, $0 < \eta \ll 1$.

is to provide a mechanism which establishes the initial scale estimate; in our case, one has to prove (**DS**.$0, N$) with a value of the "mass" $m \geq CL_0^{-1/2}$ and sufficiently large L_0. Apart from the fact that, unlike the case of strong disorder considered in [6], the value of the "mass" cannot be made arbitrarily large ($m \geq m(g) \to \infty$ as $|g| \to \infty$), the rest of the MPMSA procedure can be performed exactly in the same way as for $|g| \gg 1$.

For this reason, we only have to replace [6, Theorem 3] with Theorem 10. This theorem, combined with the arguments and results of [6], will prove Theorem 1 under assumption **(P2)**.

For application to Hamiltonians with unbounded from below random external potentials, we give the following statement, namely the Combes–Thomas estimate for large spectral gap.

Lemma 18 (Combes–Thomas for large gap)**.** *Consider a Schrödinger operator*

$$H = -\Delta + W(x)$$

acting in $\ell^2(\Lambda)$*,* $\Lambda \subset \mathbb{Z}^\nu$*,* $\nu \geq 1$*, with an arbitrary potential* $W\colon \Lambda \to \mathbb{R}$*.*

If $\operatorname{dist}(E, \sigma(H) \geq \eta \geq 2$*, then for any* $n, m \in \mathbb{Z}^\nu$*,*

$$\left|(H - E)^{-1}(n, m)\right| \leq \mathrm{e}^{-\frac{1}{2}\ln(\frac{\eta}{4\nu})|n-m|_1}. \tag{5.1}$$

Proof. The proof is obtained by a modification of that of Lemma 3: see the proof of [11, Theorem 11.2]. $\square$

Theorem 10 (initial length scale)**.** *Let* $\mathbf{H}^{(N)}(\omega) = -\boldsymbol{\Delta} + \sum_{j=1}^{N} V(x_j, \omega) + \mathbf{U}$*, where* $\boldsymbol{\Delta}$*,* $\mathbf{U}$*,* V *satisfy* (**K**)*,* (**I**)*,* (**P2**)*, respectively. Let*

$$\mathcal{E}^* := \inf\{E \in \mathbb{R} : F_V(E) > 0\}$$

where F_V *is defined by* $F_V(\lambda) = \mathbb{P}\{V(0, \omega) \leq \lambda\}$*.*

(A) *If* $\mathcal{E}^* > -\infty$*, then for any* $p > 0$*,* $C > 0$ *and arbitrary large* $L_0 > 0$ *there exists* $E^* > E^0$ *such that properties* (**S**.$0, N$) *and* (**DS**.$0, N$) *hold true in* $I = [E^0, E^*]$ *for some* $m \geq CL_0^{-1/2} > 0$*.*

(B) *If* $\mathcal{E}^* = -\infty$*, then for any* $p > 0$*,* $m > 0$ *and* $L_0 > 1$ *there exists* $E^* > -\infty$ *such that properties* (**S**.$0, N$) *and* (**DS**.$0, N$) *hold true in* $I = (-\infty, E^*]$ *with the chosen value of* $m > 0$*.*

Proof. Taking into account the Combes–Thomas estimate (Lemma 3), assertion (A) follows directly from Lemma 6 in the same way as Lemma 5 in section 3.3.

For (B), Lemma 7 and assumption $\inf\{\lambda \in \mathbb{R} : F_V(\lambda) > 0\} = -\infty$ imply that $\forall\, \epsilon > 0$ and $\forall\, \eta > 0$, $\exists\, E^*(\eta) < 0$ such that

$$\mathbb{P}\{E_0(\omega) \leq E^* + \eta\} \leq \epsilon.$$

On the other hand $E_0(\omega) > E^* + \eta$ implies that for all $E \le E^*$

$$\mathrm{dist}\big(E, \sigma(H_{\mathbf{C}_L^{(N)}(\mathbf{x})})\big) > \eta.$$

This allows to apply the above version of the Combes–Thomas estimate (Lemma 18) with $\eta \ge 2$ and even arbitrarily large, that proves $(\mathbf{S}.0, N)$, hence (B). ☐

6. Stability under weak perturbations

Now we turn to the proof of multi-particle localization for systems with weak interaction.

Like in the case of disordered systems at low energies, the principal adaptation of the multi-particle multiscale analysis developed in [6] to weakly interacting systems, without the assumption of strong disorder ($|g| \gg 1$) lies in the verification of the initial scale estimate.

Since the notions of E-resonance and (E, m)-singularity have been defined with respect to a given random ensemble of Hamiltonian, with fixed parameters, it is convenient to change now the notations and to introduce explicitly the parameters g, h of $\mathbf{H}^{(N)}_{g,h}(\omega)$. We will say that

- a cube $\mathbf{C}^{(N)}_L(\mathbf{u})$ is (E, g, h)-resonant, if it is E-resonant with respect to $\mathbf{H}^{(N)}_{g,h}(\omega)$;
- a cube $\mathbf{C}^{(N)}_L(\mathbf{u})$ is (E, m, g, h)-singular, if it is (E, m)-singular with respect to $\mathbf{H}^{(N)}_{g,h}(\omega)$.

6.1. **Proof of Theorem 2.** First of all we prove a lemma:

Lemma 19. *Suppose that the Hamiltonians $\mathbf{H}^{(N)}_{g,0}(\omega)$ (without inter-particle interaction) fulfill the following condition: for sufficiently large L_0,*

$$\mathbb{P}\left\{\exists\, E \in I : \mathbf{C}^{(N)}_{L_0}(\mathbf{u}) \text{ is } (E, m, g, 0)\text{-}S\right\} \le L_0^{-2p}, \tag{6.1}$$
$$p > 6Nd,\ m > CL_0^{-1/2},$$

and that, consequently, the MPMSA allows to derive by induction the estimates

$$\mathbb{P}\left\{\exists\, E \in I : \mathbf{C}^{(N)}_{L_k}(\mathbf{x}) \text{ and } \mathbf{C}^{(N)}_{L_k}(\mathbf{y}) \text{ are } (E, m, g, 0)\text{-}S\right\} \le L_k^{-2p}, \tag{6.2}$$

for all $k \in \mathbb{N}$ and any pair of L_k-distant cubes $\mathbf{C}_{L_k}(\mathbf{x})$, $\mathbf{C}_{L_k}(\mathbf{y})$.

Then there exists $h^* > 0$ such that for all $h \in (-h^*, h^*)$ the Hamiltonian $\mathbf{H}^{(N)}_{g,h}(\omega)$, with interaction of amplitude $|h|$, satisfies similar bounds

$$\mathbb{P}\left\{\exists E \in I : \mathbf{C}^{(N)}_{L_0}(\mathbf{u}) \text{ is } (E, m', g, h)\text{-}S\right\} \le L_0^{-2p'}, \tag{6.3}$$
$$p' > 6Nd,\ m' > CL_0^{-1/2},$$

and consequently

$$\mathbb{P}\left\{\exists\, E \in I : \mathbf{C}^{(N)}_{L_k}(\mathbf{x}) \text{ and } \mathbf{C}^{(N)}_{L_k}(\mathbf{y}) \text{ are } (E, m', g, h)\text{-}S\right\} \le L_k^{-2p'}. \tag{6.4}$$

Proof. Let us prove the assertion concerning the estimate (6.3). For simplicity, set $m^* = CL_0^{-1/2}$ with the same C, m, L_0 as in (6.1), and

$$\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}),g,h}(E) = (\mathbf{H}^{(N)}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}),g,h} - E)^{-1},\ g, h \in \mathbb{R}.$$

By definition, a cube $\mathbf{C}^{(N)}_{L_0}(\mathbf{u})$ is $(E,m,g,0)$-NS iff

$$\max_{\mathbf{y}\in\partial^-\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}\left|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}),g,0}(\mathbf{u},\mathbf{y};E)\right|\le \mathrm{e}^{-m(1+L_0^{-1/8})L_0}, \tag{6.5}$$

where, by hypothesis, $m>m^*$. Therefore, there exists $\epsilon>0$ such that

$$\max_{\mathbf{y}\in\partial^-\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}\left|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,0}(\mathbf{u},\mathbf{y};E)\right|\le \mathrm{e}^{-m^*(1+L_0^{-1/8})L_0}-\epsilon. \tag{6.6}$$

Since, by assumption, $p>6Nd$, there exist $6Nd<p'<p$ and $\delta>0$ such that $L_0^{-2p'}-\delta>L_0^{-2p}$. With such values p' and δ, inequality (6.1) with $p>6Nd$ implies

$$\mathbb{P}\{\exists E\in I:\mathbf{C}^{(N)}_{L_0}(\mathbf{u})\text{ is }(E,m,g,0)\text{-S}\}<L_0^{-2p'}-\delta. \tag{6.7}$$

Next, it follows from the second resolvent identity that

$$\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}),g,0}(E)-\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}),g,h}(E)\|\le|h|\,\|\mathbf{U}\|\cdot\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}),g,0}(E)\|\cdot\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,h}(E)\|. \tag{6.8}$$

By Lemma 2, applied to Hamiltonians[4] $\mathbf{H}^{(N)}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}),g,0}$ and $\mathbf{H}^{(N)}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}),g,h}$, for any $\delta>0$ there is $B(\delta)\in(0,+\infty)$ such that

$$\begin{aligned}&\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,0}(E)\|\ge B(\delta)\big\}\le\frac{\delta}{4}\,,\\&\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,h}(E)\|\ge B(\delta)\big\}\le\frac{\delta}{4}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,0}(E)-\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,h}(E)\|\ge|h|\,\|\mathbf{U}\|B^2(\delta)\big\}\\&\quad\le\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,0}(E)\|\ge B(\delta)\big\}+\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,h}(E)\|\ge B(\delta)\big\}\\&\quad\le 2\,\frac{\delta}{4}=\frac{\delta}{2}\,.\end{aligned}$$

Set

$$h^*:=\frac{\epsilon}{2\|\mathbf{U}\|(B(\delta))^2}>0.$$

We see that if $|h|\le h^*$, then

$$|h|\times\|\mathbf{U}\|\times(B(\delta))^2\le\frac{\epsilon}{2}\,.$$

Hence,

$$\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,0}-\mathbf{G}_{\mathbf{C}_{L_0}(\mathbf{u})g,h}\|\ge\frac{\epsilon}{2}\big\}\le 2\,\frac{\delta}{4}\,. \tag{6.9}$$

Combining (6.6), (6.8), and (6.9), we obtain

$$\begin{aligned}&\mathbb{P}\big\{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})\text{ is }(E,m',g,h)\text{-S}\big\}\\&\quad\le\mathbb{P}\big\{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})\text{ is }(E,m,g,0)\text{-S}\big\}+\mathbb{P}\big\{\|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,0}(E)-\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})g,h}(E)\|\ge\frac{\epsilon}{2}\big\}\\&\quad\le\big(L_0^{-2p'}-\delta\big)+\frac{\delta}{2}<L_0^{-2p'}.\end{aligned}$$

□

Now Theorem 2 follows from Lemma 19 combined with the assertion of Theorem 4 in which the interval $[E^0,E^*]$ is replaced by I.

[4]Note that the amplitude of the interaction has no impact on the bound (3.7).

6.2. **Proof of Corollary 1.** In this subsection, $d = 1$. In order to apply Theorem 2 to weakly disordered N-particle one-dimensional lattice Anderson models, recall that one can prove sufficiently strong initial scale estimates for the Hamiltonian

$$H(\omega) = -\Delta + gV(\omega)$$

with a non-constant i.i.d. random potential $V\colon \mathbb{Z} \times \Omega \to \mathbb{R}$ (see [3, 12]). Namely, for any $g \neq 0$ and any interval $I \subset \mathbb{R}$, for some $a = a(F_V, g) > 0$ and $m = m(F_V, g) > 0$, for all L_0 large enough

$$\mathbb{P}\left\{\exists\, E \in I : \mathrm{C}_{L_0}(u) \text{ and } \mathrm{C}_{L_0}(v) \text{ are } (E, m)\text{-S}\right\} \leq \mathrm{e}^{-aL_0}. \tag{6.10}$$

Actual proofs, which are non-perturbative in one dimension, allow to derive directly an exponential upper bound on eigenfunction correlators. Let

$$\Upsilon_{\mathrm{C}_{L_0}(u)}(x, y, \omega) = \sum_{E_j \in \sigma(H_{\mathrm{C}_{L_0}(u)})} |\psi_j(x)\psi_j(y)|\,,$$

then

$$\mathbb{E}\left[\Upsilon_{\mathrm{C}_{L_0}(u)}(x, y, \omega)\right] \leq \mathrm{e}^{-3a|x-y|}. \tag{6.11}$$

It remains to prove that for the N-particle random Hamiltonian without interaction $\mathbf{H}^{(N)}_{g,0}(\omega)$ in the one-dimensional lattice $\mathbb{Z}^1$ one has:

$$\mathbb{P}\left\{\mathbf{C}^{(N)}_{L_0}(\mathbf{u}) \text{ is } (E, m)\text{-S}\right\} \leq L_0^{-2p}.$$

To this end we first give a preliminary argument dealing with some representation of the Green functions. Namely,

$$\begin{aligned}\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E) &= \sum_{j_1, \dots, j_N} \frac{\mathbf{\Psi}_{j_1,\dots,j_N}(\mathbf{u})\mathbf{\Psi}_{j_1,\dots,j_N}(\mathbf{y})}{E - (E_{j_1} + \cdots + E_{j_N})}\\ &= \sum_{j_1, \dots, j_N} \frac{\psi_{j_1}(u_1)\psi_{j_1}(y_1)\dots\psi_{j_N}(u_N)\psi_{j_N}(y_N)}{E - (E_{j_1} + \cdots + E_{j_N})}.\end{aligned}$$

Set

$$\max_{\mathbf{y} \in \partial^- \mathbf{C}^{(N)}_{L_0}(\mathbf{u})} |\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E)| = |\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E)|$$

so that $|\mathbf{u} - \mathbf{v}| = L_0$, then there exists $j \in \{1, \dots, N\}$ such that $|u_j - v_j| = L_0$ and since all eigenfunctions are normalised, we get:

$$\begin{aligned}|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E)| &\leq \frac{|\mathbf{C}^{(N)}_{L_0}(\mathbf{u})|\Upsilon_{C_{L_0}(u_j)}(u_j, v_j; \omega)}{\operatorname{dist}(E, \sigma(\mathbf{H}^{(N)}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}))}\\ &\leq \frac{3^N L_0^N \Upsilon_{C_{L_0}(u_j)}(u_j, v_j; \omega)}{\operatorname{dist}(E, \sigma(\mathbf{H}^{(N)}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}))}.\end{aligned}$$

Now, we have that:

$$
\begin{aligned}
&\mathbb{P}\left\{\exists E\in I:\mathbf{C}^{(N)}_{L_0}(\mathbf{u})\text{ is }(E,m)\text{-S}\right\}\\
&\quad=\mathbb{P}\left\{\exists E\in I:|\mathbf{G}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}(\mathbf{u},\mathbf{v};E)|>\mathrm{e}^{-\gamma(m,L_0,N)L_0}\right\}\\
&\quad\le\mathbb{P}\left\{\exists E\in I:\frac{3^N L_0^N\Upsilon_{C_{L_0}(u_j)}(u_j,v_j;\omega)}{\operatorname{dist}(E,\sigma(\mathbf{H}^{(N)}_{\mathbf{C}^{(N)}_{L_0}(\mathbf{u})}))}>\mathrm{e}^{-\gamma(m,L_0,N)L_0}\right\}\\
&\quad\le\mathbb{P}\left\{\exists E\in I:\mathbf{C}^{(N)}_{L_0}(\mathbf{u})\text{ is }E\text{-R}\right\}\\
&\qquad+\mathbb{P}\left\{\Upsilon_{C_{L_0}(u_j)}(u_j,v_j;\omega)>3^{-N}L_0^{-N}\mathrm{e}^{-\gamma(m,L_0,N)L_0-L_0^{1/2}}\right\}\\
&\quad\le L_0^{-q}+\frac{\mathbb{E}\left[\Upsilon_{C_{L_0}(u_j)}(u_j,v_j;\omega)\right]}{3^{-N}L_0^{-N}\mathrm{e}^{-\gamma(m,L_0,N)L_0-L_0^{1/2}}}\\
&\quad\le L_0^{-q}+\mathrm{e}^{-3aL_0}3^N L_0^N\mathrm{e}^{\gamma(m,L_0,N)L_0+L_0^{1/2}}\\
&\quad\le\frac{L_0^{-2p}}{2}+\frac{L_0^{-2p}}{2},
\end{aligned}
$$

where we used property (**WS1**.N) (Wegner type bounds) to estimate the probability of E-resonance in lines 4-5 and the bound (6.11) in line 6. □

Acknowledgments

The author is grateful to Anne Boutet de Monvel and Victor Chulaevsky for the formulation of the problem, helpful discussions and encouragement.

References

[1] M. Aizenman and S. Warzel, *Localization bounds for multiparticle systems*, Commun. Math. Phys. (2009), 903–934.
[2] M. Aizenman and S. Warzel, *Complete dynamical localization in disordered quantum multi-particle systems*, XVIth International Congress on Mathematical Physics, World Sci. Publ., Hackensack, NJ, 2010, pp. 556–565.
[3] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Vol. 20, Birkhäuser Boston Inc., Boston, MA, 1990. Bound states in random media.
[4] V. Chulaevsky, *A remark on charge transfer processes in multi-particle systems*, 2010, arXiv:1005.3387.
[5] V. Chulaevsky and Y. Suhov, *Eigenfunctions in a two-particle Anderson tight binding model*, Comm. Math. Phys. **289** (2009), 701–723.
[6] V. Chulaevsky and Y. Suhov, *Multi-particle Anderson Localization: Induction on the number of particles*, Math. Phys. Anal. Geom. **12** (2009), 117–139.
[7] V. Chulaevsky and T. Spencer, *Positive Lyapunov exponents for a class of deterministic potentials*, Commun. Math. Phys **168** (1995), 455-466.
[8] H. von Dreifus and A. Klein, *A new proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **124** (1989), 285–299.
[9] T. Ekanga, *On two-particle Anderson localization at low energies*, C. R. Acad. Sci. Paris, Ser. I **349** (2011), 167–170.
[10] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, *Constructive proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **101** (1985), 21–46.
[11] W. Kirsch, *An Invitation to Random Schrödinger Operators*, 2008, Panorama et Synthèses, 25, Soc. Math. France, Paris.

[12] H. Kunz and B. Souillard, *Sur le spectre des opérateurs aux différences finies aléatoires*, Commun. Math. Phys. **78** (1985), 201–246.
[13] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer-Verlag, Berlin Heidelberg, 1992.
[14] C. Sadel and H. Schulz-Baldes, *Positive Lyapunov exponents and localization bounds for strongly mixing potentials*, : Adv. Theor. Math. Phys. **12** (2008), 1377–1399.
[15] P. Stollmann, *Caught by disorder*, Progress in Mathematical Physics, vol. 20, Birkhäuser Boston Inc., Boston, MA, 2001. Bound states in random media.

*Institut de Mathématiques de Jussieu, Université Paris Diderot, 175 rue du Chevaleret, 75013 Paris, France
E-mail address: ekanga@math.jussieu.fr